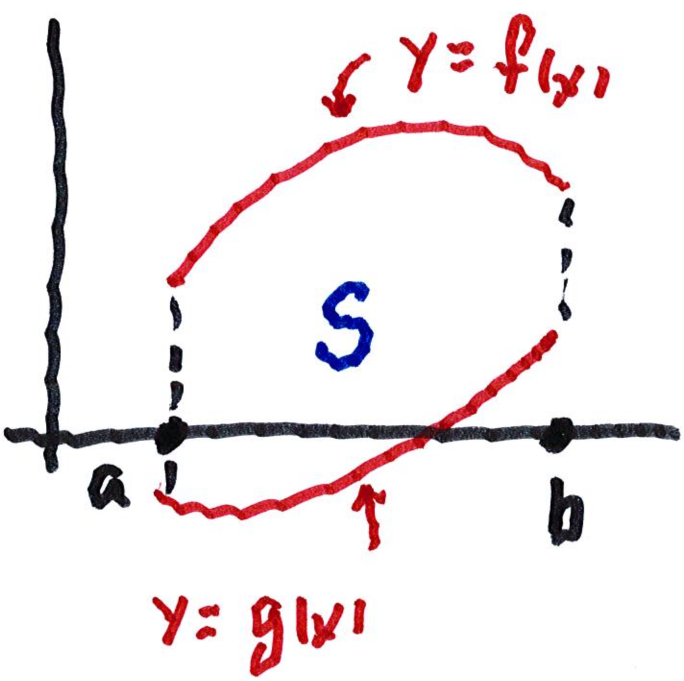
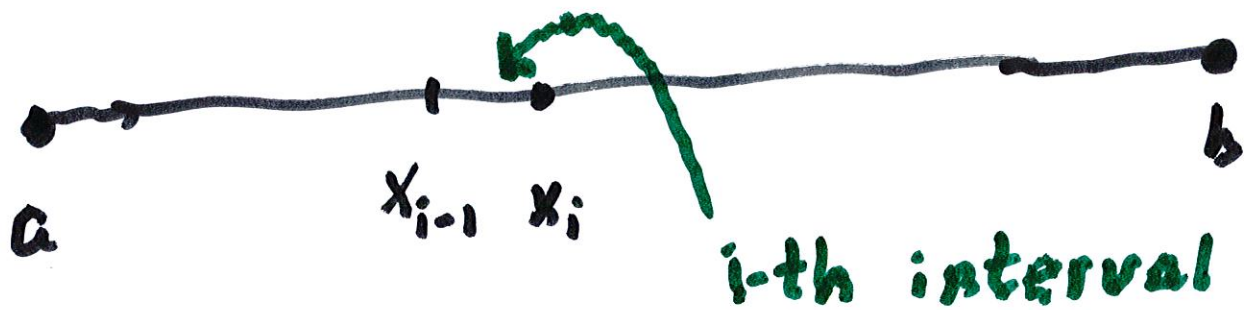
Areas between curves 6.1

Suppose $f(x) \geq g(x)$ for all x with $a \leq x \leq b$.

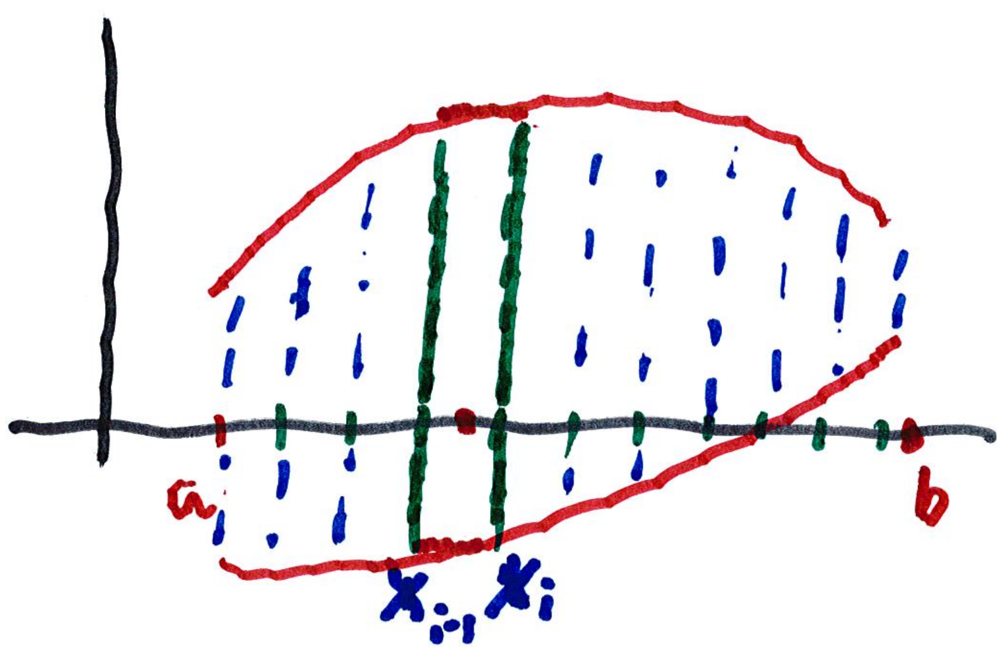
Let S be the region between the curves $y = f(x)$ and $y = g(x)$, and between the lines $x = a$ and $x = b$



We divide $[a, b]$ into n intervals of size $\frac{b-a}{n} = \Delta x$



We can approximate S by n strips.



Let x_i^* be any random point in $[x_{i-1}, x_i]$. The height of the i-th strip is approximately $\left(f(x_i^*) - g(x_i^*)\right)$

The area is $\approx \left(f(x_i^*) - g(x_i^*)\right)\Delta x$

The total area of all n strips is

$$A_n = \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

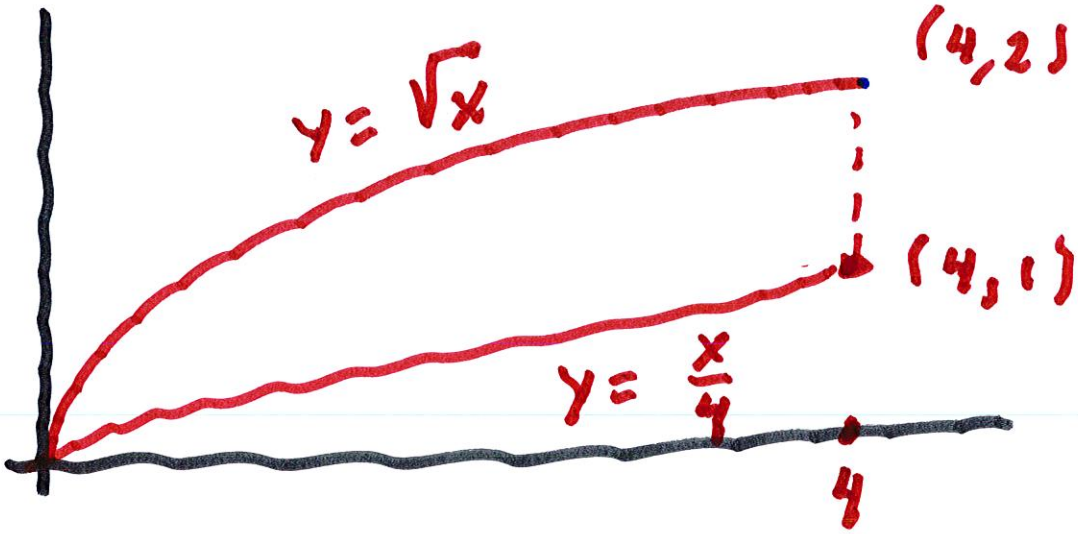
Assuming f, g are continuous,

as $n \to \infty$,

$$A = \lim_{n\to\infty} A_n = \int_a^b \{f(x) - g(x)\}\,dx$$

We define A to be the area of S.

Ex. Let S be the region bounded above by $y = \sqrt{x}$ and below by $y = \frac{x}{4}$ for $0 \le x \le 4$.

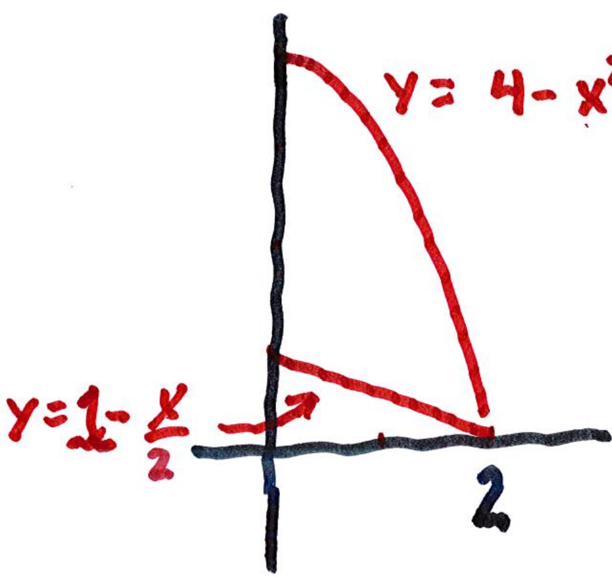


Then the area $A = \int_0^4 \sqrt{x} - \frac{x}{4}\, dx$

$$= \frac{2}{3} x^{3/2} - \frac{x^2}{8} \Big|_0^4$$

$$= \frac{2}{3} 4^{3/2} - \frac{4^2}{8} = \frac{16}{3} - 2 = \frac{10}{3}$$

Ex. Let S be the region bounded between $y = 4 - x^2$, $y = 1 - \frac{x}{2}$, and $x = 0$. Find the area of S



$y(x) = 4 - x^2$ is the top curve,

$y(x) = 1 - \frac{x}{2}$ is the bottom curve.

$$A = \int_0^2 (4 - x^2) - (1 - \frac{x}{2})\, dx$$

Where do curves coincide?

$$4 - x^2 = 1 - \frac{x}{2}$$

$$0 = x^2 - \frac{x}{2} - 3$$

$$\therefore \quad x = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 12}}{2} = \frac{\frac{1}{2} \pm \sqrt{\frac{49}{4}}}{2}$$

$$\therefore \quad x = \frac{1 \pm 7}{4} = \underline{\underline{2}} \quad \text{or} \quad x = \underline{\underline{-\frac{3}{2}}}$$

$$\therefore \quad A = \int_0^2 3 + \frac{x}{2} - x^2 \quad dx$$

$$= \int_0^2 \left(3 + \frac{x}{2} - x^2\right) dx$$

$$= 3x + \frac{x^2}{4} - \frac{x^3}{3} \Big|_0^2$$

$$= 6 + 1 - \frac{8}{3} = \frac{13}{3}$$

In general, if $y_T(x)$ is the top curve and $y_B(x)$ the bottom curve, then

Area of $S = \int_a^b (y_T - y_B)\, dx$.

Ex. Find the area of the region enclosed by $y = x^2$ and $y = x + 6$.

We need to find the endpoints a and b. We set

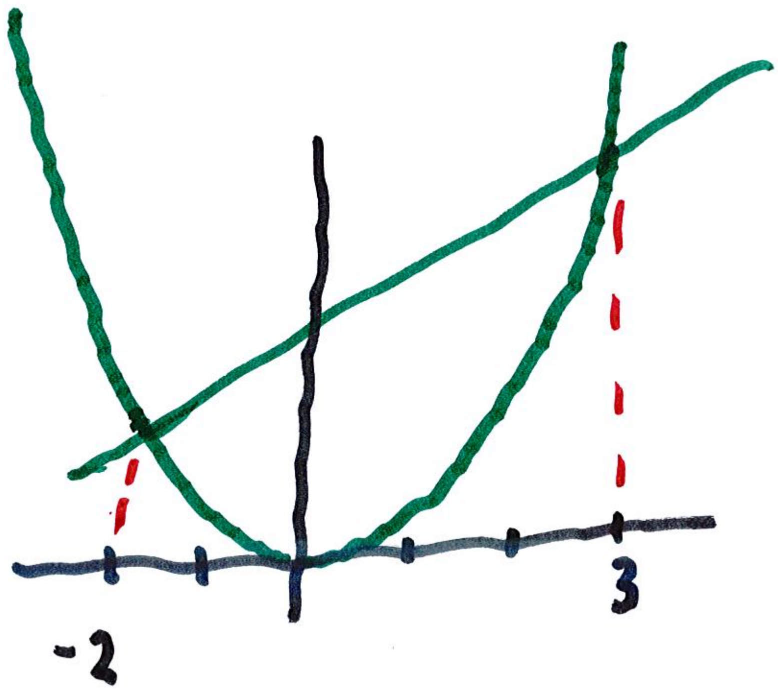
$$x^2 = x + 6$$

$$\rightarrow \quad x^2 - x - 6 = 0 .$$

or $(x-3)(x+2) = 0$

so $x = 3$ or $x = -2$

$\therefore$ $a = -2$, $b = 3$



$y = x+6$ is a line and $y = x^2$ is a parabola curved upward.

$\therefore$ $y_T = x+6$ and $y_B = x^2$

$$\therefore \text{Area} = \int_{-2}^{3} (x + 6 - x^2)\, dx$$

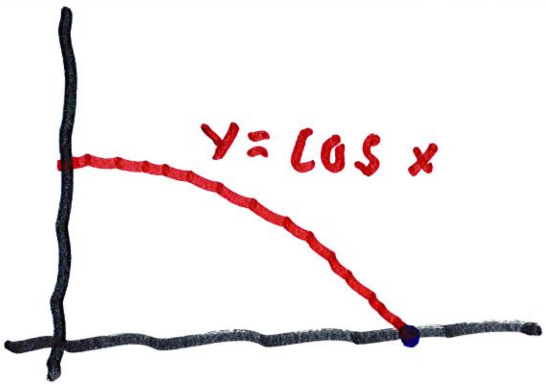
$$= \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right)$$

$$= \left(\frac{9}{2} + 9\right) - \left(-\frac{22}{3}\right)$$

$$= \frac{81}{6} + \frac{44}{6} = \frac{125}{6}$$

Ex. Find the area under the curve $y = \cos x$, for $0 \le x \le \frac{\pi}{2}$



In this case,

$y_T(x) = \cos x$

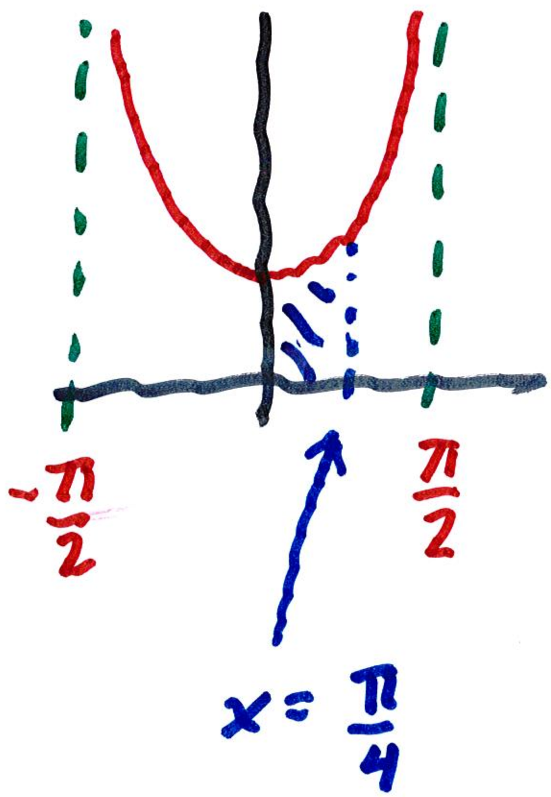
and $y_B(x) = 0$

$$A = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$$

$$= 1 - 0 = \underline{\underline{1}}$$

Ex. Find the area of the region bounded by $y = \dfrac{1}{\cos^2 x}$ and $x = \dfrac{\pi}{4}$ in the first quadrant.

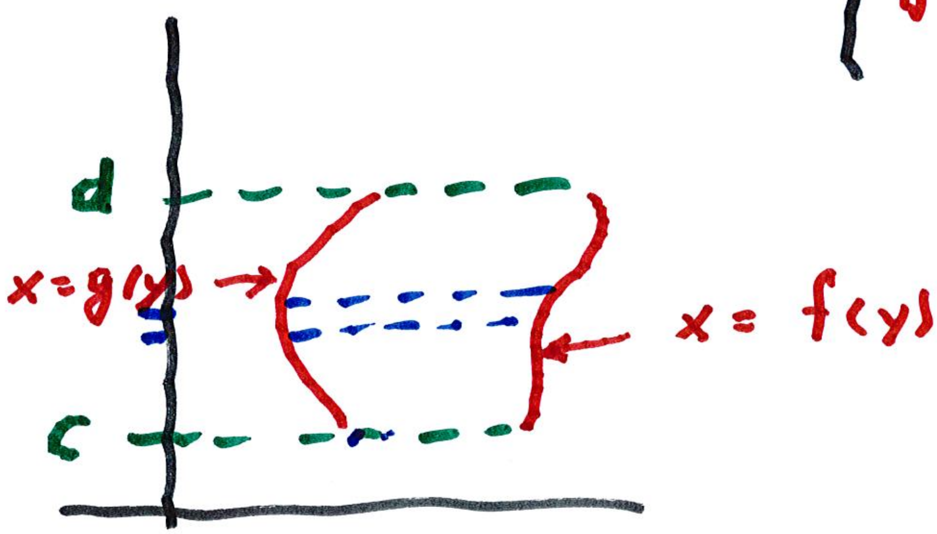
$$\frac{1}{\cos^2 x} = \sec^2 x$$



$$A = \int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_0^{\pi/4}$$

$$= 1 - 0 = \underline{\underline{1}}$$

Sometimes it's better to view a region as $\begin{cases} c \le y \le d \\ g(y) \le x \le f(y) \end{cases}$



$$A = \int_c^d \left(f(y) - g(y) \right) dy$$

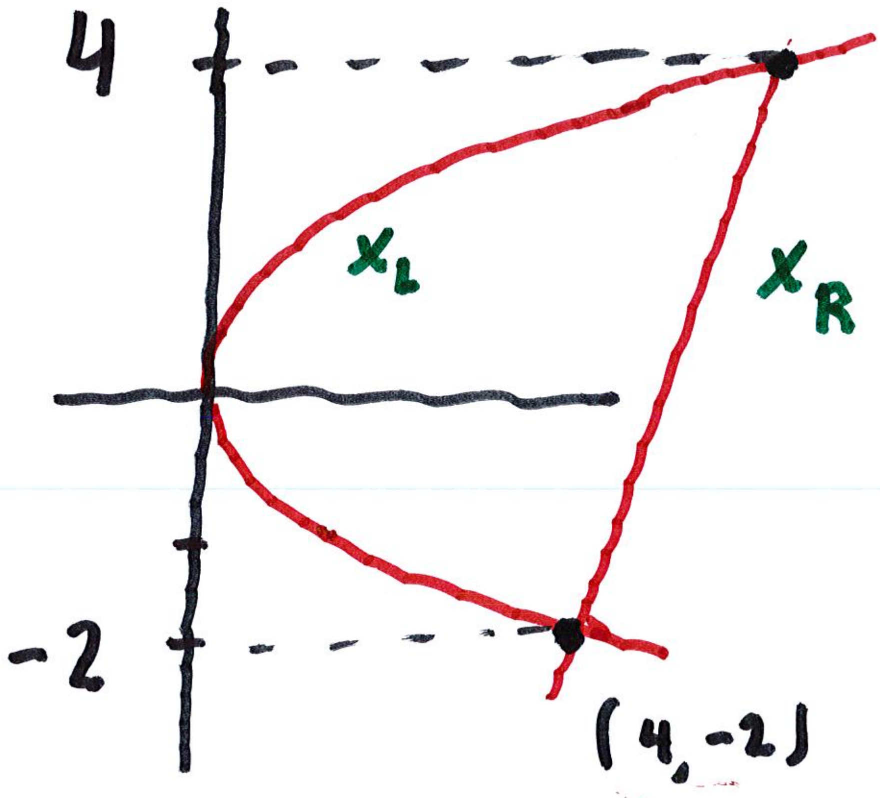
Ex. Find the area of the region bounded by $x = y^2$ and $x = 2y + 8$.

$\rightarrow \; y^2 = x = 2y + 8$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$\rightarrow \; y = -2$ or 4



$$A = \int_{-2}^{4} x_R - x_L \, dx$$

$$A = \int_{-2}^{4} \left((2y + 8) - y^2 \right) dy$$

$$= y^2 + 8y - \frac{y^3}{3} \Bigg|_{-2}^{4}$$

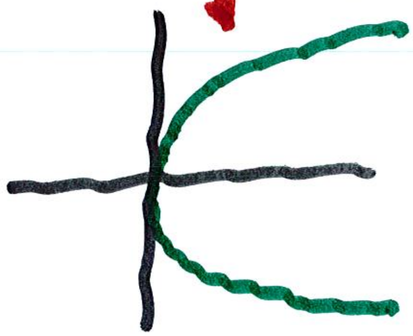
$$= \left(16 + 32 - \frac{64}{3}\right)$$

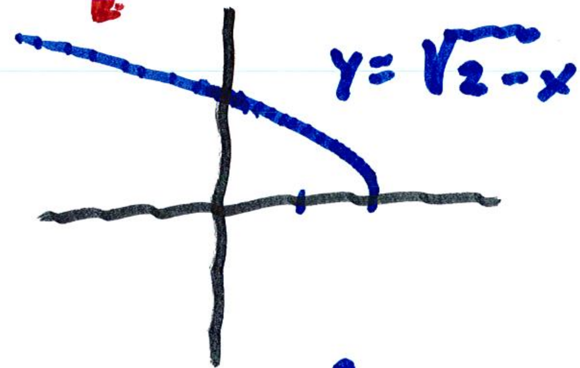
$$- \left(4 - 16 + \frac{8}{3}\right)$$

$$= \frac{80}{3} - \left(-\frac{28}{3}\right) = \frac{108}{3} = \underline{\underline{36}}$$

Ex Find the area of the region bounded by

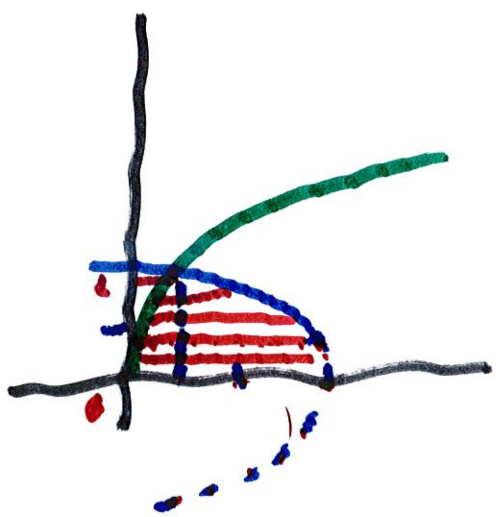
$x = y^4$, $\quad y = \sqrt{2-x}$, $\quad$ and $y = 0$





$y^2 = 2 - x$

$\rightarrow \; x = 2 - y^2$



$y^4 = x = 2 - y^2$

$y^4 + y^2 - 2 = 0$

$(y^2 + 2)(y^2 - 1) = 0$

$\therefore \; y = 1$

Note $x_R = 2 - y^2$

and $x_L = y^4$

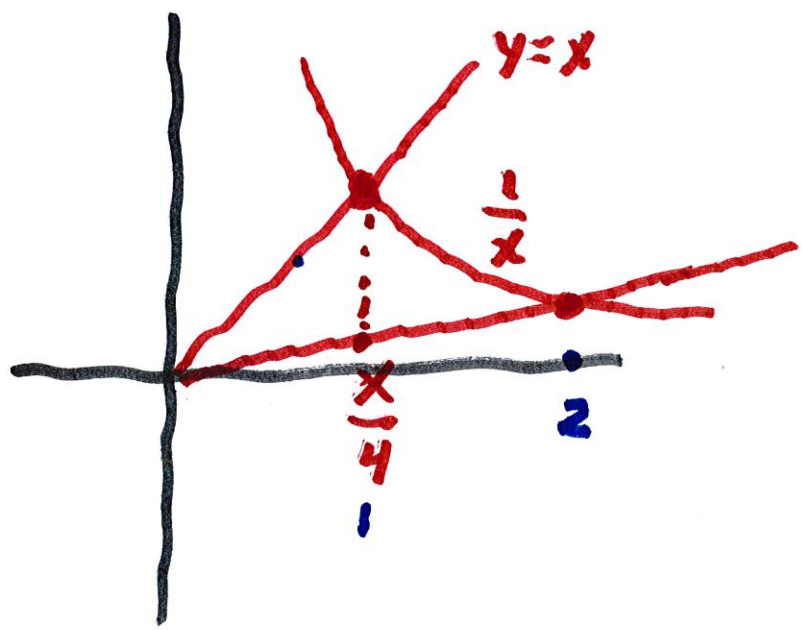
$$\therefore A = \int_0^1 (2-y^2) - y^4 \quad dy$$

$$= 2y - \frac{y^3}{3} - \frac{y^5}{5} \Big|_0^1$$

$$= 2 - \frac{1}{3} - \frac{1}{5} = \frac{22}{15}$$

Sometimes one must split the region into 2 regions.

Ex. Find the area of the region bounded by $y = \frac{1}{x}$, $y = x$, and $y = \frac{x}{4}$, with $x > 0$



$y = x \ , \quad y = \frac{1}{x} \quad \rightarrow \quad x = \frac{1}{x}$

or $\quad x^2 = 1 \rightarrow x = 1$

$\frac{1}{x} = \frac{x}{4} \rightarrow x^2 = 4$

$x = \pm 2$

Also, $\quad \frac{1}{x} = \frac{x}{4} \quad \rightarrow \quad 4 = x^2 \rightarrow x = 2$

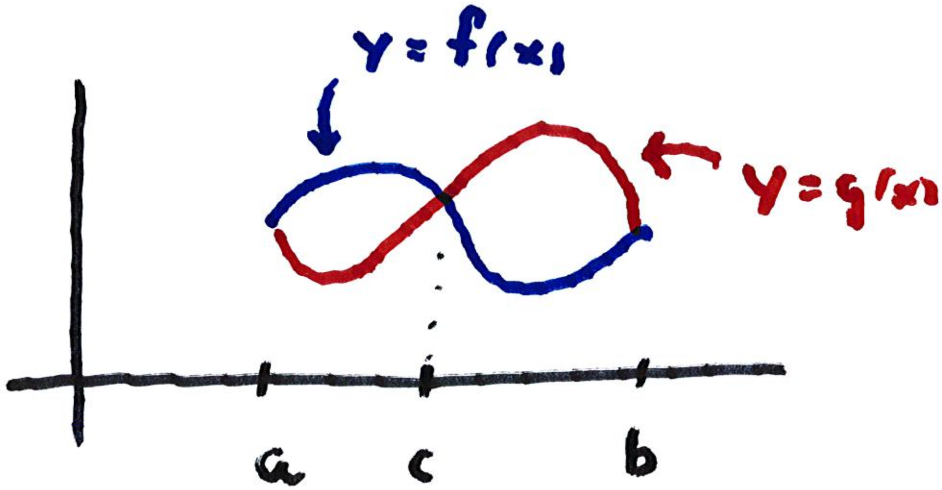
$$\therefore \text{Area} = \int_0^1 \left(x - \frac{x}{4} \right) dx$$

$$+ \int_1^2 \left(\frac{1}{x} - \frac{x}{4} \right) dx$$

$$= \int_0^1 \frac{3x}{4}\,dx + \left(\ln x - \frac{x^2}{8}\right)\Bigg|_1^2$$

$$= \frac{3}{8} + \left(\ln 2 - \frac{1}{2}\right) - \left(0 - \frac{1}{8}\right)$$

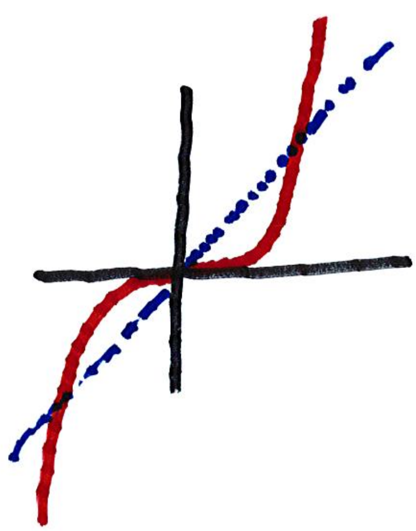
$$= \ln 2$$



Find area of region enclosed by

$y = f(x)$ and $y = g(x)$:

Ex. $y = x^3$, $y = x$



$x^3 = x$

$\rightarrow x(x^2 - 1) = 0$

$\rightarrow x = 0, 1, -1$

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

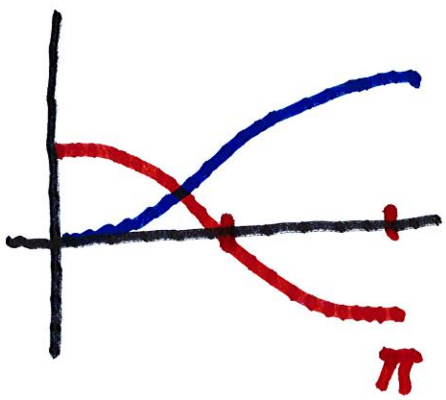
$$= \frac{x^4}{4} - \frac{x^2}{2} \Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^4}{4} \Big|_{0}^{1}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{1}{4} = \underline{\underline{\frac{1}{2}}}$$

Ex. Area of region enclosed by

$y = \cos x \quad , \quad y = 1 - \cos x$



They're equal when

$\cos x = 1 - \cos x$

$\rightarrow \cos x = \frac{1}{2}$

$$A = \int_0^{\frac{\pi}{3}} \cos x - (1-\cos x)\, dx$$

$$+ \int_{\frac{\pi}{3}}^{\pi} (1-\cos x - \cos x)\, dx$$

$$= 2\sin x - x \Big|_0^{\frac{\pi}{3}} + (x - 2\sin x)\Big|_{\frac{\pi}{3}}^{\pi}$$

$$= \left(\sqrt{3} - \frac{\pi}{3}\right) + (-1)\left(\frac{\pi}{3} - \sqrt{3}\right)$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$