6.2 Volumes

Volume of cylinder is \( V = \pi R^2 h \)

More generally, a generalized cylinder

Then \( V = Ah \)

\[ B = \text{base} \]
\[ A = \text{area of base} \]
Now consider a region like an unsliced loaf of bread:

\[ A(x) = \text{area of } x\text{-cross section} \]

where \( a \leq x \leq b \).
Now set $\Delta x = \frac{b-a}{n}$.

with

$$a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$
Volume of slice between $x_{i-1}$ and $x_i$ is $\approx A(x_i) \Delta x$

Volume of all $n$ slices is

$\approx V = \sum_{i=1}^{n} A(x_i) \Delta x$

As $n \to \infty$

$V = \int_{a}^{b} A(x) dx$
Ex. Find volume of sphere of radius $R$

$$x^2 + y^2 + z^2 = R^2$$

The $x$-slice of sphere is

$$y^2 + z^2 = R^2 - x^2$$
The radius \( r = \sqrt{R^2 - x^2} \)

\[ A(x) = \pi \left( \sqrt{R^2 - x^2} \right)^2 \]

\[ A \left( x_1 \right) = \pi \left( R^2 - x^2 \right) \]

So \( V = \int_{-R}^{R} \pi \left( R^2 - x^2 \right) \, dx \)

\[ = 2\pi \int_{0}^{R} \left( R^2 - x^2 \right) \, dx \]

\[ \text{even} \]
\[ = 2\pi \left( R^2 x - \frac{x^3}{3} \right) \bigg|_0^R \]

\[ = 2\pi \left( R^3 - \frac{R^3}{3} \right) = \frac{4\pi R^3}{3} \]

Ex. Consider the region bounded by \( y = x^{\frac{1}{3}}, \ y = 0, \) and \( x = 1. \) What is volume of solid obtained by revolving it around \( x\)-axis?
The dotted line generates a disk of radius $R_x = x^{1/3}$.

$A(x) = \pi x^{2/3}$

$\text{Vol} = \left[ \pi x^{2/3} \, dx \right]_0^1 = \pi \cdot \frac{3}{5} x^{5/3} \bigg|_0^1 = \frac{3\pi}{5}$
Sometimes, we use $A_{\text{cys}} = \text{area of } y\text{-cross-section}.$

Ex. Let $D =$ region bounded by $y = \ln x, \ y = 1, \ y = 0.2$ and $x = \Delta$

What is volume obtained by rotating $D$ around $y\text{-axis}$?
The segment at \( y \) generates a disk of radius \( x = e^y \).

\[
A(y) = \pi x^2 = \pi (e^y)^2 = \pi e^{2y}
\]

So,
\[
V = \int_1^2 \pi e^{2y} \, dy
\]

\[
= \left. \frac{\pi}{2} e^{2y} \right|_1^2 = \frac{\pi}{2} (e^4 - e^2)
\]
So far we've used the disk method. Now we use the washer method.

Ex. Find the volume of region obtained by revolving

\( D \), region bounded by

\( y = 2x \) and \( y = x^2 \)

about the x-axis.
If $x^2 = 2x$, then $x = 0$ or $x = 2$

The dotted segment generates a ring. The inside radius = $x^2$ and outer radius = $(2x)$.
Area of ring

is \( \pi (2x)^2 - \pi (x^2)^2 = A(x) \)

or \( A(x) = \pi (4x^2 - x^4) \)

\[
\text{Vol} = \int_0^2 \pi \left( 4x^2 - x^4 \right) \, dx
\]

\[
= \pi \left( \frac{4x^3}{3} - \frac{x^5}{5} \right) \bigg|_0^2
\]

\[
= \pi \left( \frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi}{15}
\]
Ex. Let D be bounded by

\[ y = 2 - x, \quad y = x^2, \quad \text{and} \quad x = 0 \]

Now revolve D about \( y = -1 \)

Find volume.
The outer radius (for fixed $x$) is $(2-x) - (-1) = 3-x$.

The inner radius (for fixed $x$) is $x^2 - (-1) = x^2 + 1$.

Note $x^2 = 2-x \Rightarrow x^2 + x - 2 = 0$

$01 (x+2)(x-1) = 0 \Rightarrow x = 1$

$$Vol = \int_0^1 \pi (3-x)^2 - \pi (x^2+1)^2 \, dx$$
\[ = \pi \int_{0}^{1} \left(-x^4 + x^2 - 2x^2 - 6x + 8\right) \, dx \]

\[ = \pi \left( -\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \right) \bigg|_{0}^{1} \]

\[ = \pi \left( -\frac{1}{5} - \frac{1}{3} + 5 \right) = \frac{67\pi}{15} \]

Ex. Let D in \( \mathbb{R}^2 \) be bounded

by \( y = \sqrt{x}, \ y = 0, \ \text{and} \ x = \ 4 \)
What is volume if we revolve

D about the line $x = 5$?

Outer radius for fixed $y$ is

O.r. = $5 - y^2$ and

i.r. = 1

inner radius
\[ \text{Vol} = \int_0^2 \pi (5-y^2)^2 - \pi \cdot 1^2 \, dy \]

\[ = \pi \int_0^2 (24 - 10y^2 + y^4) \, dy \]

\[ = \pi \left[ 24y - \frac{10y^3}{3} + \frac{y^5}{5} \right]_0^2 \]

\[ = \pi \left( 48 - \frac{80}{3} + \frac{32}{5} \right) \]

\[ = \pi \left( \frac{316}{15} \right) \]

\[ \approx \frac{316}{15} \pi \]
Now let $D$ be the region

$$a \leq x \leq b, \quad g(x) \leq y \leq f(x)$$

Suppose we rotate $D$ about the line $y = A$. Find the volume.
The inner radius is

\[ \text{In. Rad} = g(x) - A \]

The outer radius is

\[ \text{O. Rad} = f(x) - A \]

\[ V = \int_{a}^{b} \pi \left( f(x) - A \right)^2 - \pi \left( g(x) - A \right)^2 \, dx \]
Suppose $D$ is
\[
\{ \quad c \leq y \leq d \\
g(y) \leq x \leq f(y) \}
\]

Rotate $D$ about $x = B$

I.R. = $B - f(y)$

O.R. = $B - g(y)$
\[ V = \pi \int_a^d \left( (B - g(y))^2 - (B - f(y))^2 \right) dy \]

Ex. The base of a solid \( S \) is the region bounded by

\[ y = \sqrt{x} \text{ and } y = \frac{x}{2}. \]

Cross sections perpendicular to the \( x \)-axis are squares.

Find the volume of \( S \).
\[
\sqrt{x} = \frac{x}{2}
\]
\[
x = \frac{x^2}{4}
\]
\[
4x = x^2
\]
\[\rightarrow x = 0 \text{ or } x = 4
\]

Side of square is
\[
\sqrt{x} - \frac{x}{2}
\]
Vol. = \int_0^4 \left( \sqrt{x} - \frac{x}{2} \right)^2 \, dx

= \int_0^4 x - x^{3/2} + \frac{x^2}{4} \, dx

= \left[ \frac{x^2}{2} - \frac{2}{5} x^{5/2} + \frac{x^3}{12} \right]_0^4

= 8 - \frac{64}{5} + \frac{16}{3}

= \frac{8}{15}
Other solids:

Ex. Let $D$ = region in plane $z = 0$ that is bounded by $y = x$ and $y = x^2$.

Suppose $S$ is a solid with base $D$ such that each cross-section perpendicular to the $x$-axis is a square.
side of square is \( (x-x^2) \)

\[ \therefore A(x) = (x-x^2)^2 \]

\[ \Rightarrow Vol = \int_0^1 (x-x^2)^2 \, dx \]

\[ = \left( \frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right) \bigg|_0^1 = \]

\[ \]
\[
\frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}
\]

Now suppose \( D \) is bounded by \( y = \sqrt{1-x^2} \) and \( y = 0 \), and that the \( x \)-cross-sections \( \perp \) to base \( D \) are equilateral triangles.
Area of Equi-\(\Delta\)

\[
\text{is } A = \frac{\sqrt{3} S}{4}
\]

\[
\therefore \text{If } S = \sqrt{1 - x^2}, \text{ then}
\]

\[
A(x) = \frac{\sqrt{3} (1 - x^2)}{4}
\]

\[
\therefore V = \int_{-1}^{1} \frac{\sqrt{3} (1 - x^2)}{4} \ dx = \frac{\sqrt{3}}{3}
\]