6.3 Volumes by Cylindrical Shells

Suppose \( D \) is the set of \((x, y)\) such that \( a \leq x \leq b \) and \( 0 \leq y \leq f(x) \). (\( a > 0 \))

Rotate \( D \) about the \( y \)-axis.
to get a solid $S$. What is the volume of $S$?

Consider a shell with outside radius $r_2$, inside radius $r_1$, and height $h$.

The volume of the shell is

$$V = V_2 - V_1$$
\[ \pi n_2^2 h - \pi n_1^2 h \]
\[ = \pi \left( n_2^2 - n_1^2 \right) h \]
\[ = \pi \left( n_2 + n_1 \right) \left( n_2 - n_1 \right) h \]
\[ = 2\pi \frac{n_2 + n_1}{2} h \left( n_2 - n_1 \right). \]

We let \( \Delta n = n_2 - n_1 \)

\( = \) thickness of shell

and \( \bar{n} = \) average radius.
The volume of a cylindrical shell is

\[ V = 2\pi rh \Delta n \]

or also

\[ V = \text{[circumference]} \times \text{[height]} \]

\[ \text{[thickness]} \]

Now subdivide \([a, b]\) by

\[ a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_n = b \]
we get (approximately) a cylindrical shell with

volume \( V_i = 2\pi \bar{x}_i f(x_i) \Delta x \),

\( \bar{x}_i = \text{midpoint of } [x_{i-1}, x_i] \)
The total volume generated by all strips is

\[ V = \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x \]

If \( n \to \infty \), then \( \Delta x \to 0 \), and

\[ V = \int_{a}^{b} 2\pi x f(x) \, dx \]

Note 2\(\pi\)x = circum. of shell
\( f(x) \) = height of shell
\( dx \) = thickness
Ex. Let $D$ be defined by

$$D \leq x \leq 1, \quad 0 \leq y \leq x^3,$$

and rotate about $y$-axis.

Find volume.

$$V = \int_0^1 2\pi x \cdot x^3 \, dx$$
\[
\int_0^1 2\pi x^4 \, dx = \frac{2\pi x^5}{5} \bigg|_0^1 = \frac{2\pi}{5}
\]

Ex. Let \( D \) be the region under \( y = e^{x^2} \) for \( 0 \leq x \leq 1 \). What is the volume if we rotate \( D \) about the y-axis?
\[ V = \int_0^1 2\pi x \cdot e^{x^2} \, dx \]

Make substitution \( u = x^2 \)

\[ \Rightarrow du = 2x \, dx \quad x = 0 \Rightarrow u = 0 \]
\[ x = 1 \Rightarrow u = 1 \]
\[
\int_0^1 \pi e^{x^2} 2x \, dx
\]

\[
= \pi \left[ e^{x^2} \right]_0^1 = \pi e - \pi
\]

Now suppose \( D \) is defined by
\[
a \leq x \leq b, \quad g(x) \leq y \leq f(x)
\]
\[ \text{Vol} = \int_{a}^{b} 2\pi x \left( f(x) - g(x) \right) \, dx \]

Ex. Find the volume of the solid obtained by rotating

\[ D = \{ (x, y); \quad 0 \leq x \leq 2, \quad \frac{x^2}{2} \leq y \leq 4 - x \} \]

about the y-axis.
\[ V = \int_0^2 2\pi x \left( 4 - x - \frac{x^2}{2} \right) \, dx \]

\[ = \pi \int_0^2 8x - 2x^2 - x^3 \, dx \]

\[ = \pi \left. \left( 4x^2 - \frac{2x^3}{3} - \frac{x^4}{4} \right) \right|_0^2 \]

\[ = \pi \left( 16 - \frac{16}{3} - 4 \right) \]

\[ = \pi \cdot \frac{20}{3} \]
Ex. Use cyl. shells to compute vol. if \( D \), which is bounded by \( x = 1 + (y-2)^2 \) and \( x=2 \) is rotated about \( x\)-axis.

Find endpoints:

\[ 2 = 1 + (y-2)^2 \]

\[ \rightarrow 1 = (y-2)^2 \]

\[ y=1 \text{ or } 3 \]

\[ \rightarrow \pm 1 = y-2 \]
\[ V = \int_{1}^{3} 2\pi y \left( 2 - (1 + (y-2)^2) \right) dy \]

\[ = 2\pi \int_{1}^{3} y \left( 1 - (y-2)^2 \right) dy \]

\[ = 2\pi \int_{1}^{3} y \left( -3 + 4y - y^2 \right) dy \]
\[ 2\pi \int_1^3 (-3y + 4y^2 - y^3) \, dy \]

\[ = 2\pi \left( \left. -\frac{3y^2}{2} + \frac{4y^3}{3} - \frac{y^4}{4} \right|_1^3 \right) \]

\[ = 2\pi \left( \left( -\frac{27}{2} + 36 - \frac{81}{4} \right) - \left( -\frac{3}{2} + \frac{4}{3} - \frac{1}{4} \right) \right) \]

\[ = 2\pi \left( -12 + \frac{104}{3} - 20 \right) \]

\[ = 2\pi \cdot \frac{8}{3} = \frac{16\pi}{3} \]
\[
\begin{align*}
&= 4\pi \int_{-1}^{1} (1 - x - x^2 + x^3) \, dx \\
&= 4\pi \int_{0}^{1} (1 - x^2) \, dx \\
&= 8\pi \left( x - \frac{x^3}{3} \right) \bigg|_{0}^{1} \\
&= 8\pi \left( \frac{2}{3} \right) = \frac{16\pi}{3}
\end{align*}
\]
Find volume if we rotate D about $x = 1$

$x^2 = 2 - x^2$

$\Rightarrow 2x^2 = 2 \Rightarrow x = 1, -1$

$$Vol = \int_{-1}^{1} 2\pi (1-x)(2-x^2-x^2) \, dx$$

$$= 4\pi \int_{-1}^{1} (1-x)(1-x^2) \, dx$$
How do we decide whether to integrate in $x$ or in $y$?

Ex. Suppose $D$ is defined by $y = F(x)$ and $y = G(x)$ for $a \leq x \leq b$.

Revolve about $x$-axis.
Apply Washer Method:

\[ V_{ol} = \int_{a}^{b} \pi F(x)^2 - \pi G(x)^2 \, dx \]

Now suppose we want to use the Shell Method:

\[ V_{ol} = \int_{c}^{d} 2\pi y (f(y) - g(y)) \, dy \]
Which method we use depends on whether it's easier to solve for $x$ in $y = F(x)$ and $y = G(x)$.

**Other Regions:** Suppose $D$ is bounded by $y = \sqrt{1-x^2}$ and $y = 0$. 
and that $x$-cross sections \perp to base $D$ are equilateral triangles:

Suppose the side of $\Delta$ is $s = 5$

Area of $\Delta$ is $A = \frac{\sqrt{3} \cdot s^2}{4}$
If \( s = \sqrt{1-x^2} \), then \( A(x) = \frac{\sqrt{3} (1-x^2)}{4} \).

\[
\text{Vol} = \int_{-1}^{1} \frac{\sqrt{3} (1-x^2)}{4} = \frac{\sqrt{3}}{3}.
\]