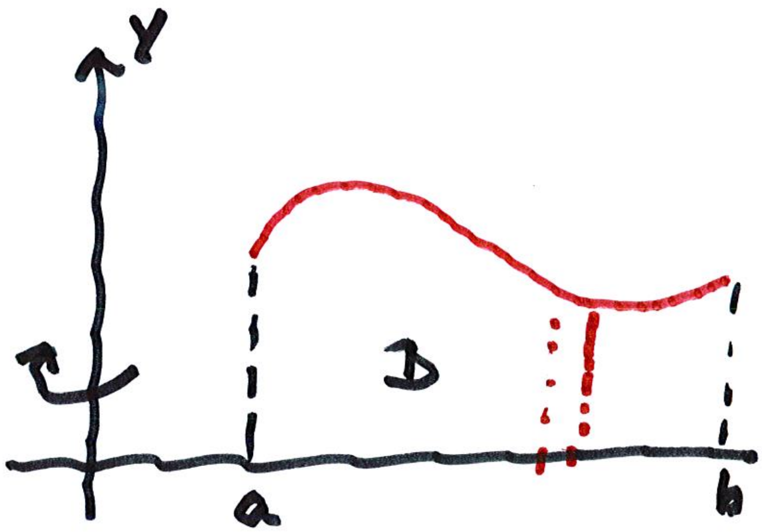
6.3 Volumes by Cylindrical Shells

Suppose D is the set of

(x, y) such that $a \le x \le b$

and $0 \le y \le f(x)$. $(a > 0)$

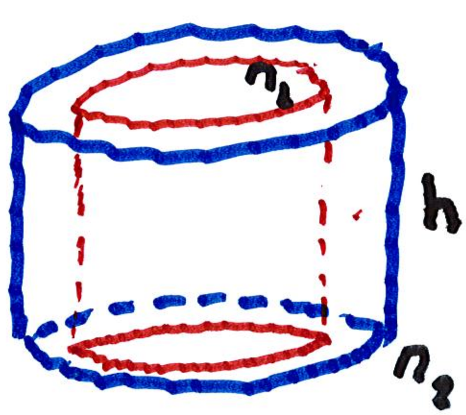


Rotate D about the y-axis

to get a solid S. What is the volume of S?

Consider a shell with outside radius r_2, inside radius r_1 and height h.

The volume of the shell is



$$V = V_2 - V_1$$

$$= \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi (r_2^2 - r_1^2) h$$

$$= \pi (r_2 + r_1)(r_2 - r_1) h$$

$$= 2\pi \frac{r_2 + r_1}{2} h (r_2 - r_1).$$

We let $\Delta r = r_2 - r_1$

= thickness of shell

and r = average radius.

The volume of a cylindrical shell is

$$V = 2\pi r h \Delta r$$

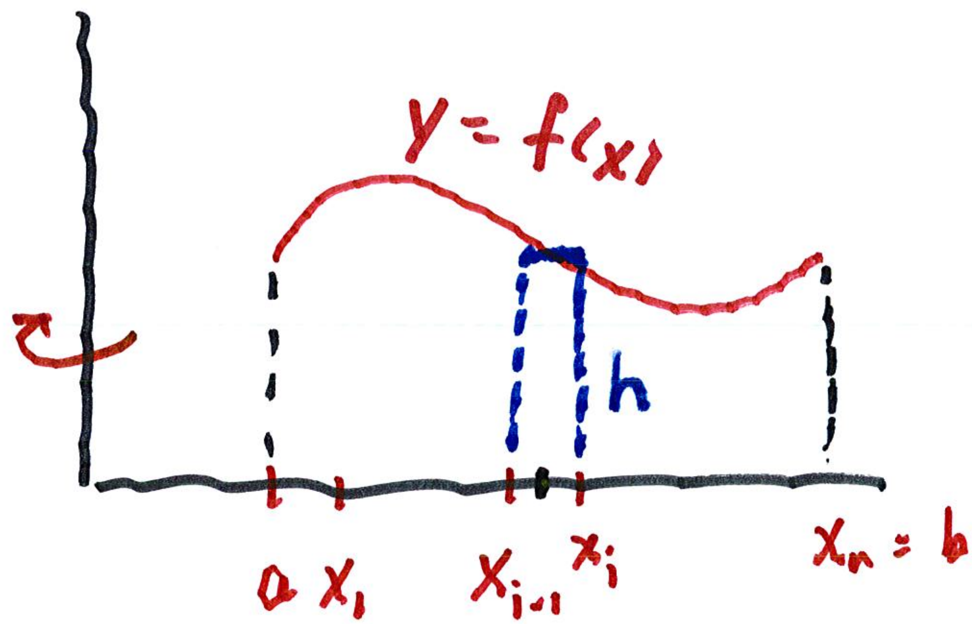
or also

$$V = [\text{circumference}][\text{height}]$$

$$[\text{thickness}]$$

Now subdivide $[a, b]$ by

$$a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots x_n = b$$



we get (approximately) a

cylindrical shell with

volume $V_i = 2\pi \bar{x}_i f(\bar{x}_i)\Delta x$,

$\bar{x}_i$ = midpoint of $[x_{i-1}, x_i]$

The total volume generated by all strips is

$$V = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$

If $n \to \infty$, then $\Delta x \to 0$, and

$$V = \int_a^b 2\pi x f(x)\, dx$$

Note $2\pi x$ = circum. of shell

$f(x)$ = height of shell

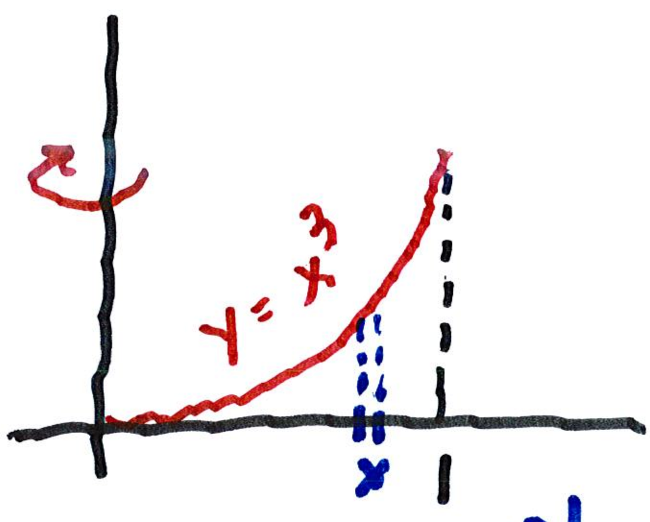
dx = thickness

Ex. Let D be defined by

$0 \le x \le 1, \quad 0 \le y \le x^3,$

and rotate about y-axis.

Find volume.

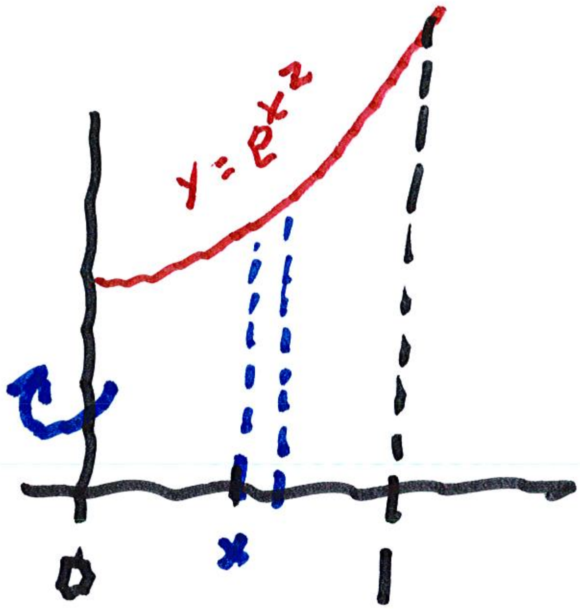


$$V = \int_0^1 2\pi x \cdot x^3 \, dx$$

$$= \int_0^1 2\pi x^4 \, dx = \frac{2\pi x^5}{5} \Big|_0^1$$

$$= \frac{2\pi}{5}$$

Ex. Let D be the region under $y = e^{x^2}$ for $0 \le x \le 1$. What is the volume if we rotate D about the y-axis?



$$V = \int_0^1 2\pi x \cdot e^{x^2}\, dx$$

Make substitution $u = x^2$

$\rightarrow du = 2x\,dx \qquad x = 0 \rightarrow u = 0$

$\qquad\qquad\qquad\qquad x = 1 \rightarrow u = 1$

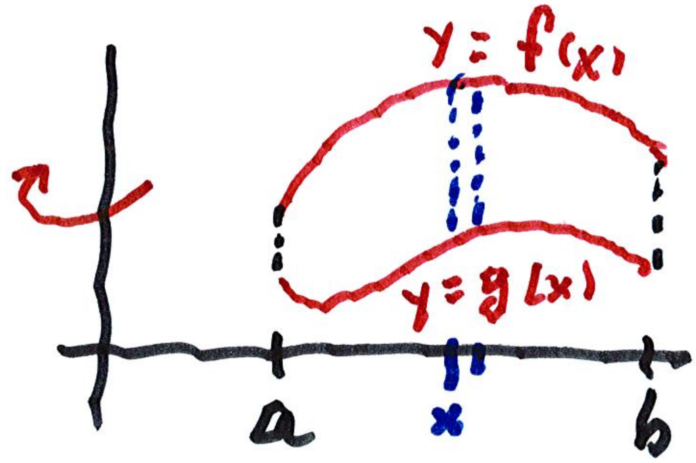
$$= \int_0^1 \pi e^{x^2} 2x \, dx$$

$$= \pi \int_0^1 e^u \, du = \pi e^u \Big|_0^1$$

$$= \underline{\underline{\pi e - \pi}}$$

Now suppose D is defined by

$a \le x \le b$, $g(x) \le y \le f(x)$

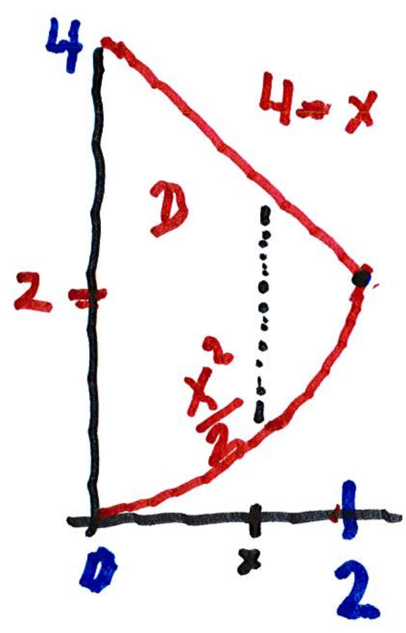


$$\text{Vol} = \int_a^b 2\pi x \, (f(x) - g(x)) \, dx$$

Ex. Find the vol. of solid obtained by rotating

$$D = \left\{ (x,y) ; \begin{array}{l} 0 \le x \le 2 \\ \frac{x^2}{2} \le y \le 4 - x \end{array} \right\}$$

about the y-axis



$$V = \int_0^2 2\pi x \left(4 - x - \frac{x^2}{2}\right) dx$$

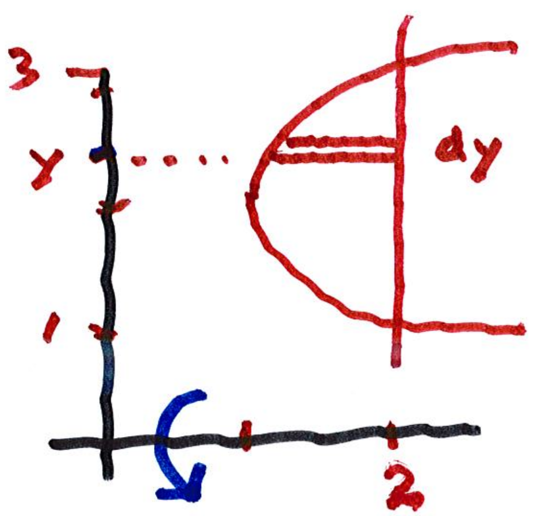
$$= \pi \int_0^2 8x - 2x^2 - x^3 \; dx$$

$$= \pi \left(4x^2 - \frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2$$

$$= \pi \left(16 - \frac{16}{3} - 4\right)$$

$$= \pi \cdot \frac{20}{3}$$

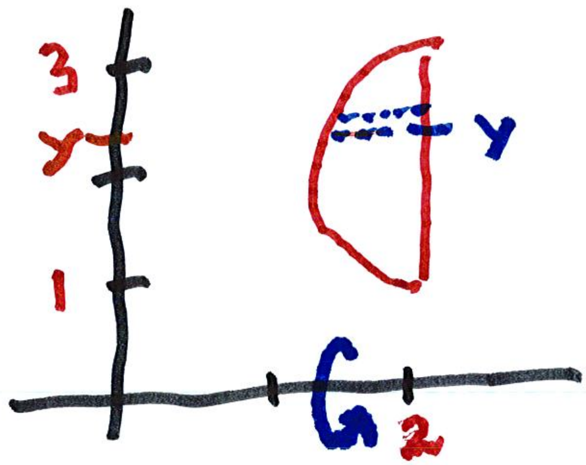
Ex. Use cyl. shells to compute vol. of D, which is bounded by $x = 1 + (y-2)^2$ and $x = 2$ is rotated about x-axis.



Find endpoints: $2 = 1 + (y-2)^2$

$\rightarrow 1 = (y-2)^2$

$y = 1$ or 3 $\quad \rightarrow \pm 1 = y - 2$



$$V = \int_1^3 2\pi y \left(\underset{\uparrow \atop x_R}{2} - \left(1 + \underset{\uparrow \atop x_L}{(y-2)^2}\right)\right) dy$$

$$= 2\pi \int_1^3 y \left(1 - (y-2)^2\right) dy$$

$$= 2\pi \int_1^3 y \left(-3 + 4y - y^2\right) dy$$

$$2\pi \int_1^3 -3y + 4y^2 - y^3 \; dy$$

$$= 2\pi \left(-\frac{3y^2}{2} + \frac{4y^3}{3} - \frac{y^4}{4} \right) \Bigg|_1^3$$

$$= 2\pi \left(-\frac{27}{2} + 36 - \frac{81}{4} \right)$$

$$- 2\pi \left(-\frac{3}{2} + \frac{4}{3} - \frac{1}{4} \right)$$

$$= 2\pi \left(-12 + \frac{104}{3} - 20 \right)$$

$$= 2\pi \cdot \frac{8}{3} = \frac{16\pi}{3}$$

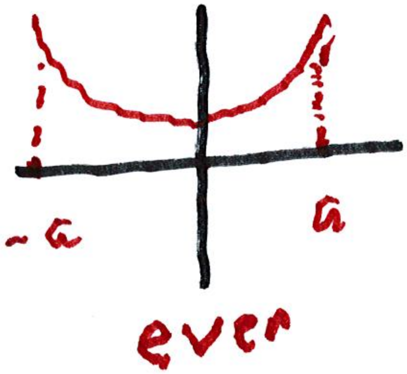
$$= 4\pi \int_{-1}^{1} (1 - x - x^2 + x^3)\,dx$$

odd

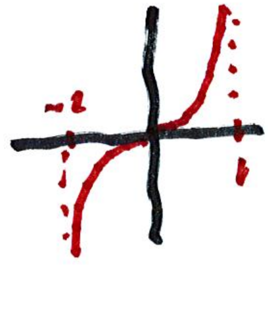
$$= 8\pi \int_{0}^{1} (1 - x^2)\,dx$$

$$= 8\pi \left(x - \frac{x^3}{3} \right) \Big|_{0}^{1}$$

$$= 8\pi \left(\frac{2}{3} \right) = \frac{16\pi}{3}$$

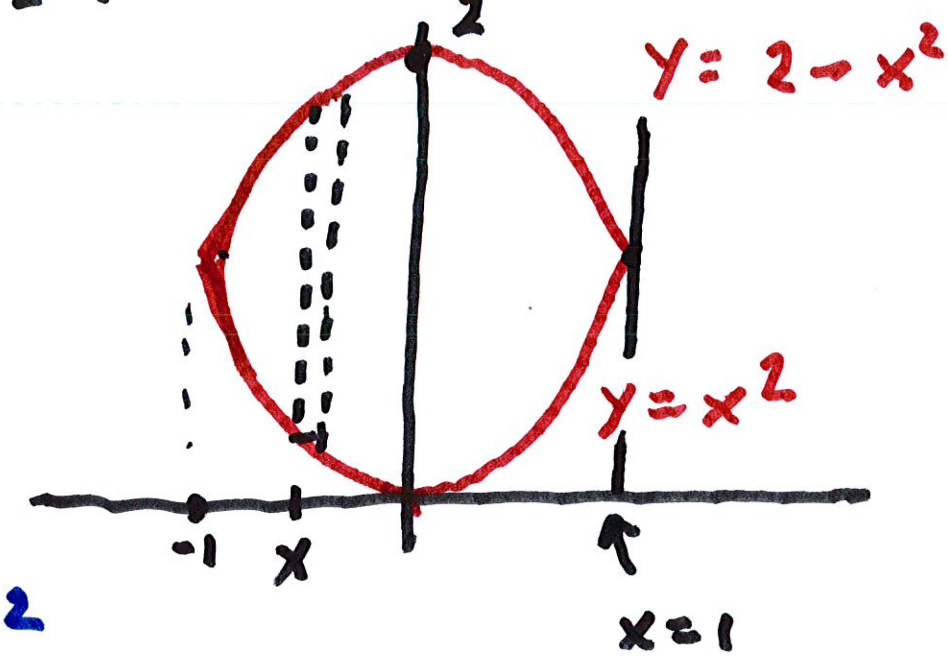


even



Find volume if we rotate D

about $x = 1$



$x^2 = 2 - x^2$

$\rightarrow 2x^2 = 2 \rightarrow x = 1, -1$

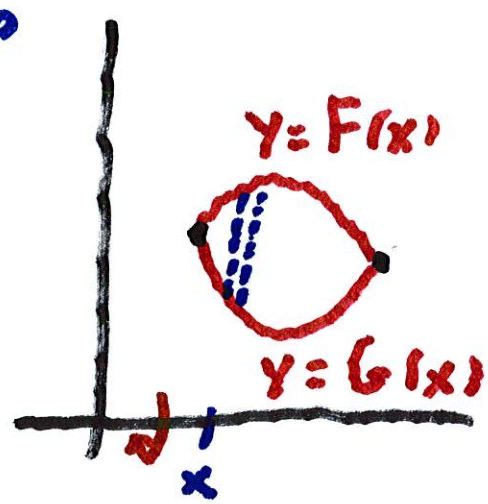
$$\text{Vol} = \int_{-1}^{1} 2\pi (1-x)(2 - x^2 - x^2)\, dx$$

$$= 4\pi \int_{-1}^{1} (1-x)(1-x^2)\, dx$$

How do we decide whether to integrate in x or in y?

Ex. Suppose D is defined by $y = F(x)$ and $y = G(x)$ for $a \le x \le b$

Revolve about x-axis

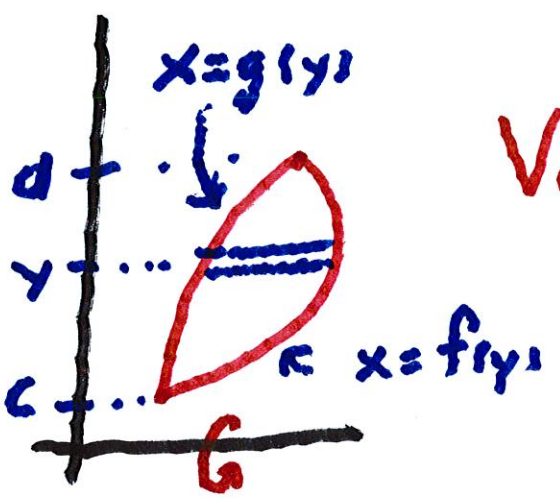


Apply Washer Method:

$$\text{Vol} = \int_a^b \pi f(x)^2 - \pi G(x)^2 \, dx$$

Now suppose we want

to use the Shell Method:

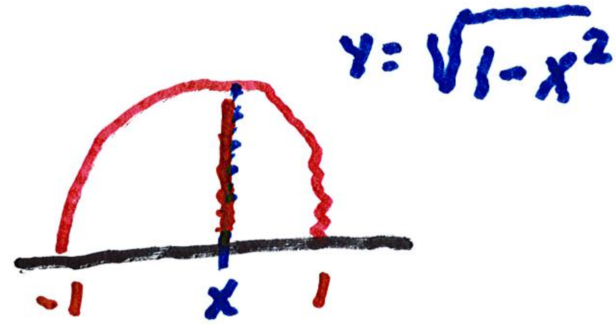


$$\text{Vol} = \int_c^d 2\pi y \big(f(y) - g(y) \big) \, dy$$

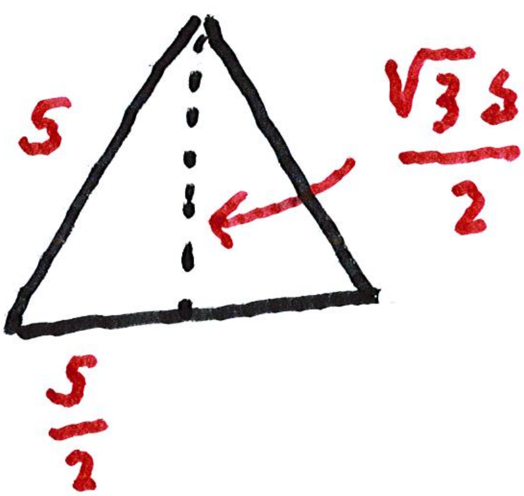
Which method we use depends on whether it's easier to solve for x in $y = F(x)$ and $y = G(x)$.

Other Regions: Suppose D is bounded by $y = \sqrt{1-x^2}$ and $y = 0$

and that x-cross sections $\perp$ to base D are equilateral triangles:



Suppose the side of Δ is $= s$



$\therefore$ Area of Δ is $A = \dfrac{\sqrt{3}\, s^2}{4}$

$\therefore$ If $s = \sqrt{1-x^2}$,

then $A(x) = \dfrac{\sqrt{3}\,(1-x^2)}{4}$

$$\therefore \quad \text{Vol} = \int_{-1}^{1} \frac{\sqrt{3}\,(1-x^2)}{4} = \frac{\sqrt{3}}{3}.$$