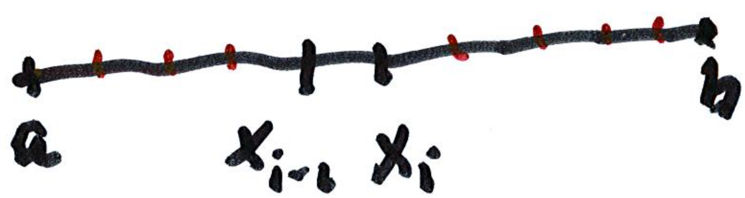
Exam on Thursday Evening

at 6:30 PM (Feb. 4th)

Covers up to Section 7.1

6.4 Work

For motion along a line, suppose the force $F = F(x)$ depends on x.



Motion from a to b.

(F doesn't change much on a small interval.)

$\therefore$ When particle moves

from x_{i-1} to x_i, the work W_i is

$$W_i \approx F(x_i^*)\Delta x,$$

where x_i^* is in interval.

$\therefore$ Total work W is

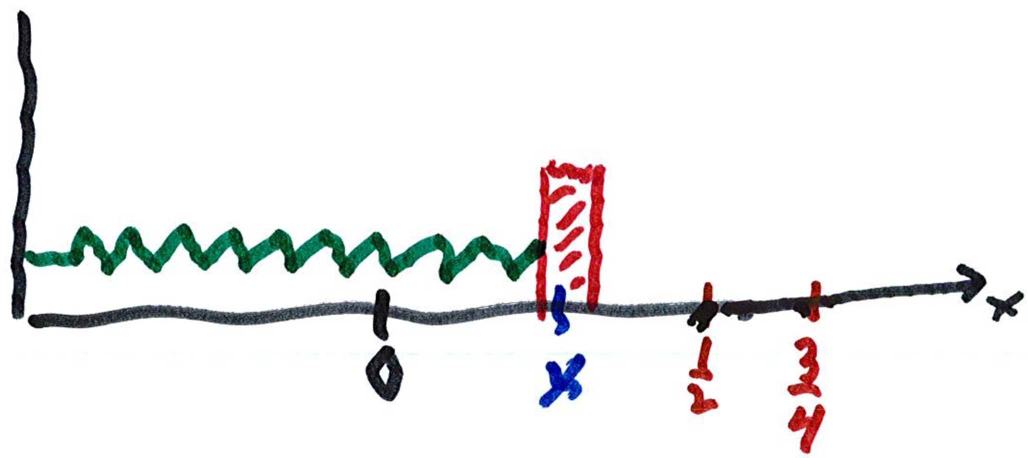
$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

As $n \to \infty$,

$$W = \int_a^b f(x)\,dx$$

Springs: Hooke's Law says that the force required to stretch a spring x units beyond its natural length is proportional to x:

$$f(x) = kx$$



natural length $(x=0)$

Ex. Suppose it takes 2N

to stretch a spring $\frac{1}{4}$ m

(past the nat. length)

How much work is needed to

stretch the spring from

$\frac{1}{2}$ m to $\frac{3}{4}$ m ?

$f(x) = kx \qquad 2 = k \cdot \left(\frac{1}{4}\right)$

$\Rightarrow k = 8$

$$\therefore \text{Work} = W = \int_{\frac{1}{2}}^{3/4} 8x \, dx$$

$$= 4x^2 \Big|_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= 4\left(\frac{3}{4}\right)^2 - 4\left(\frac{1}{2}\right)^2$$

$$= \frac{9}{4} - 1 = \frac{5}{4} \text{ J}$$

Joules

Ex. Suppose it takes 10 ft-lb. of work to stretch a spring $\frac{1}{2}$ foot from its natural position. How much work is needed to stretch it an additional $\frac{1}{2}$ ft?

First, find k:

$$10 = \int_0^{1/2} kx\,dx = \frac{kx^2}{2}\Bigg|_0^{\frac{1}{2}}$$

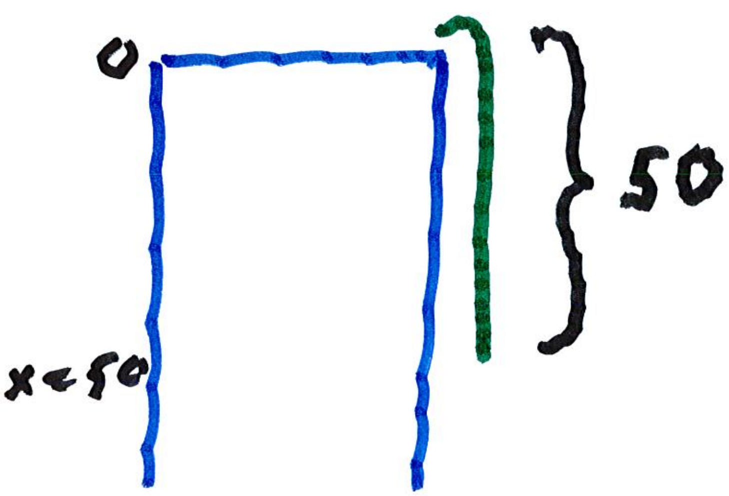
$$= \frac{k}{8} \qquad \therefore\ k = 10\cdot 8 = 80$$

Then

$$W = \int_{\frac{1}{2}}^{1} 80x\,dx = 40x^2\Bigg|_{\frac{1}{2}}^{1}$$

$$= 40 - 40(1/4) = \underline{\underline{30}} \text{ ft.lb.}$$

Ex. A cable weighing 150 lb is 50 ft long. If it's hanging from a tall building, how much work is needed to lift it to the top?



Choose x-coord. so $x=0$ is the top of bldg.

Choose a short piece

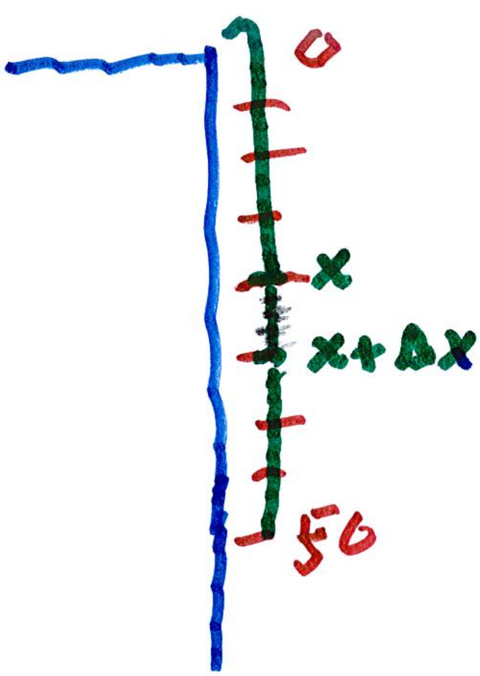
at x that is Δx long.

The linear density is

$150/50 = 3$

The short piece

weighs $3\Delta x$ lb.



To lift it to top

we need $\Delta W = \underline{x \cdot 3\Delta x}$

$\therefore$ Since the whole cable

consists of

n pieces of length Δx,

the total work is

$$W \approx \sum_{i=1}^{n} 3x_i \, \Delta x \rightarrow \int_0^{50} 3x \, dx$$

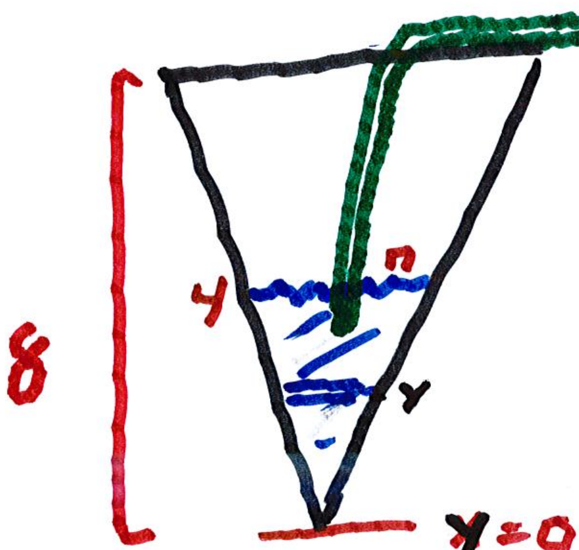
$$= \frac{3x^2}{2} \Big|_0^{50} = \frac{3}{2}(2500)$$

$$= \underline{\underline{3750 \text{ ft-lb.}}}$$

Pumping Water from a Tank

Suppose a tank is 8 m high

in the shape of a cone

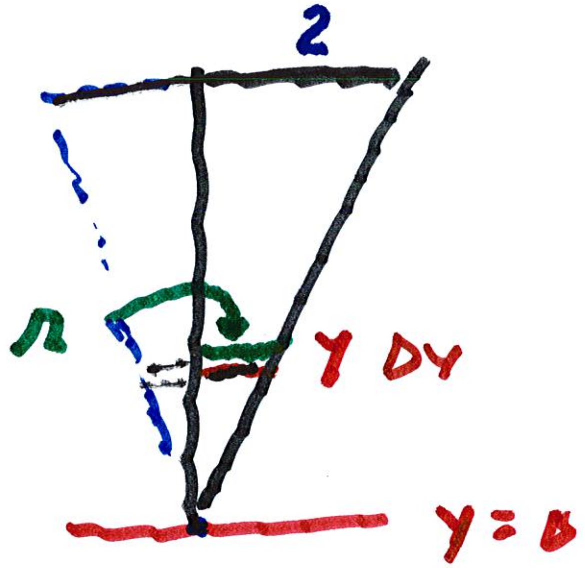


The diameter at the top is 4m, so the radius is 2 m.

Choose coord. so $x=0$ is the vertex and $x=8$ is the top.

If the water in the tank is 4 m high, how much work is needed to pump it all out?

Think of a layer between y and $y + \Delta y$



Clearly $r = ky$

$2 = k \cdot 8$

or $k = \frac{1}{4}$

$r = \frac{1}{4} y$

$$\therefore r = \frac{y}{4}$$

Cross-section at y

is a circle with

area $= \pi r^2 = \pi \left(\frac{y}{4}\right)^2$

Volume of Δy layer is

$$\Delta V = \frac{\pi y^2}{16} \Delta y \quad m^3$$

Since 1 m^3 has mass 1000 kg, the force of this layer is

$$\left(\frac{\pi y^2}{16}\Delta y\right)\cdot 9800 \text{ N}$$

This layer must be lifted $(8-y)$ m, so the work is $(8-y)\frac{\pi y^2}{16}\cdot 9800\,\Delta y$

(for that layer)

The total needed work is

$$W = \int_0^4 \frac{9800\pi}{16}\left(8y^2 - y^3\right) dy$$

$$= \frac{9800\pi}{16}\left(\frac{8}{3}y^3 - \frac{y^4}{4}\right)\Big|_0^4$$

$$= \frac{9800\pi}{16}\left(\frac{8}{3}64 - 64\right)$$

$$= 9800\pi \cdot \frac{5}{3} \cdot 4$$

$$= \underline{\frac{196{,}000\pi}{3}} \text{ J}$$

#16 A bucket weighs 4 lb and holds 40 lb. of water

We want to lift it from a well that is 80 ft deep.

Assume it's lifted at 2 ft/second and that the bucket leaks water at .2 lb/s. Calculate work W.

It takes 40 seconds to lift the bucket. After t seconds, the combined weight is $= 40 + 4 - (.2t)$

pounds.

In a short Δt

time interval, the bucket

is lifted $2\Delta t$ ft.

$\therefore$ In that Δt interval,

$$\Delta W = \left(44 - (.2t)\right) \cdot 2\Delta t$$

$$W = \int_0^{40} (44 - .2t) \cdot 2\,dt$$

$$= \left(88 - (.4t)\right)\Delta t$$

$\therefore$ Total Work is

$$W = \int_0^{40} \left(88 - (.4t)\right) dt$$

$$= 88t - (.2t^2) \Big|_0^{40}$$

$$= 3520 - 320 = 3200$$

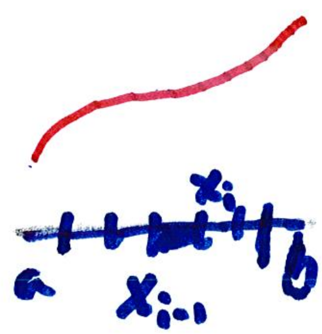
ft-lb.

6.5 Average of a Function

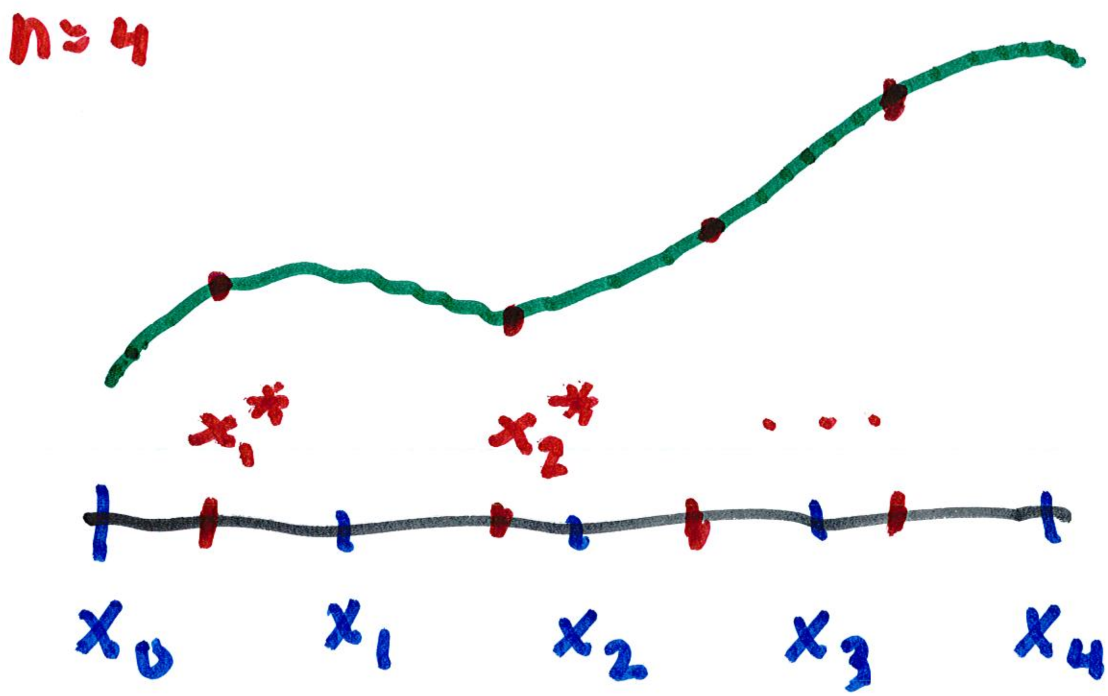
Consider a fcn. $f(x)$ over an interval. Suppose $[a,b]$ is divided into intervals

$$a = x_0 < x_1 \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

with $\Delta x = \dfrac{b-a}{n}$.



Choose x_i^* in each interval $[x_{i-1}, x_i]$



The average of n values is

$$\frac{f(x_1^*) + f(x_2^*) \dots + f(x_n^*)}{n}$$

Note $n = \dfrac{b-a}{\Delta x}$

$$= \frac{f(x_1^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}}$$

$$= \frac{1}{b-a}\left[f(x_1^*)\Delta x + \cdots + f(x_n^*) \Delta x \right]$$

$$= \frac{1}{b-a} \cdot \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\longrightarrow \frac{1}{b-a}\int_a^b f(x)\,dx$$

as $n \to \infty$

We define

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(the average value of f on $[a,b]$)

Ex. Let $f(x) = x + x^2$ on $[1,3]$

$$\int_1^3 (x+x^2)\,dx = \frac{x^2}{2} + \frac{x^3}{3}\Bigg|_1^3$$

$$= \left(\frac{9}{2} + 9\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = \frac{38}{3}$$

$$f_{avg} = \frac{19}{3} \leftarrow$$

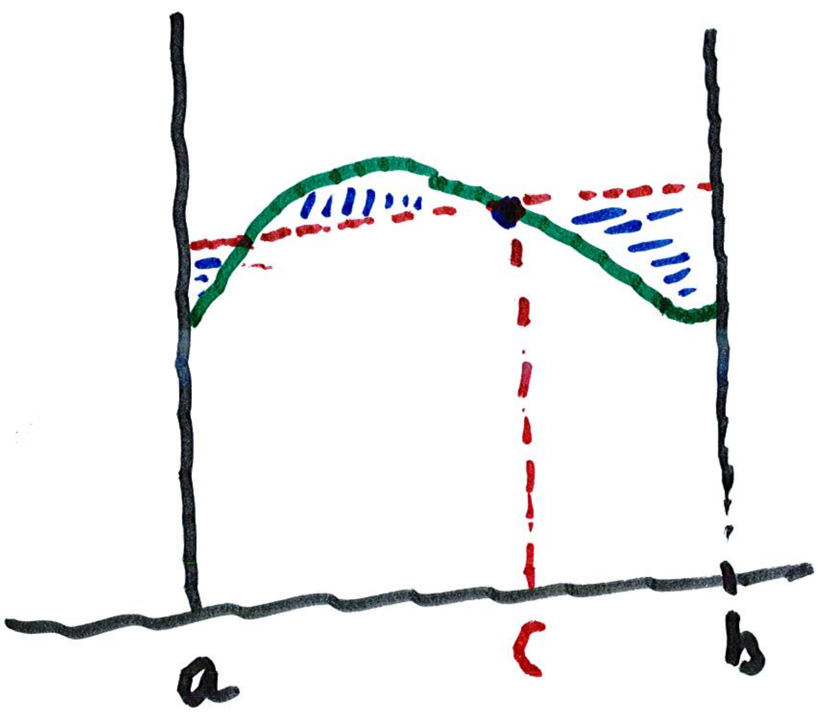
Thm. <u>Mean Value Thm. for Integrals.</u>

Suppose f is continuous on $[a, b]$. Then there is a number c in $[a, b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

That is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$



Same above f(c) as below.

Ex. The linear density of an 8 m long is $12\sqrt{1+x}$.

Calculate the average density of the rod.

Total Mass $= \int_0^8 12\sqrt{x+1}\,dx$

$= 12 \cdot \frac{2}{3}(x+1)^{3/2} \Big|_0^8$

$= 8 \cdot 9^{3/2} - 8 = 8 \cdot 27 - 8 = 208$

$\therefore$ avg. density $= \frac{208}{8} = 26 \frac{kg}{m}$

If $f(x) = x + x^2$,

$$= \int_1^3 f(x)\,dx$$

$$c + c^2 = f(c) = \frac{1}{2} \cdot \frac{38}{3} = \frac{19}{3}$$

$$\therefore \quad c = \frac{-1 + \sqrt{1 + \frac{76}{3}}}{2}$$

$$c \approx \frac{-1 + \sqrt{27}}{2} \approx \frac{-1 + 5.2}{2} = 2.1$$