Substitution Rule:

Compute $\int \sqrt{x^2+4}\ x\,dx$

Set $u = x^2 + 4 \quad du = 2x\,dx$

$$= \frac{1}{2}\int (x^2+4)^{1/2}\, 2x\,dx$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}\,u^{3/2}$$

$$= \frac{1}{3}(x^2+4)^{3/2} + C$$

7.1 Integration by Parts

Note that

$$\int \frac{d}{dx}(h(x))\,dx = h(x) + C$$

The Product Rule says

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

Hence, if we integrate:

$$\int \frac{d}{dx}(f(x)g(x)) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

$$\shortparallel$$

$$f(x)g(x)$$

$$\boxed{\therefore \int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

We can write this in an easier way

Let $u = f(x)$ and $v = g(x)$

Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

The above formula becomes

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex. Compute $\int x \sin x \, dx$

Set $u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$\therefore \int x \sin x \, dx = uv - \int v \, du$$

$$= -x\cos x - \int (-\cos x) \, dx$$

$$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$$= \underline{\underline{-x\cos x + \sin x + C}}$$

Ex. Compute $\int \ln x \, dx$

Set $u = \ln x \qquad dv = dx = 1 \cdot dx$

$du = \frac{1}{x} dx \qquad v = x$

Hence $\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$

$$= \underline{\underline{x \ln x - x + C}}$$

Ex. Sometimes we have to integrate more than once:

Compute $\int x^2 \cos x \, dx$

Set $u = x^2$ $\quad dv = \cos x \, dx$

$du = 2x\,dx \quad v = \sin x$.

$$\int x^2 \cos x \, dx = x^2 \cdot \sin x - \int \sin x \; 2x \, dx$$

$$= x^2 \sin x + \int \sin x \,(-2x)\, dx$$

Now set $u = -2x \quad dv = \sin x \, dx$

$du = -2\,dx \quad v = -\cos x$

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x$$

$$- \int \cos x \cdot 2\, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Ex. Compute $\int x^2 e^{3x}\,dx$

We want to choose u so that du is "simpler" than u and so that dv is not too much more complicated.

Set $u = x^2$ $\quad dv = e^{3x}\,dx$

$du = 2x\,dx \quad v = \frac{e^{3x}}{3}$

$$\therefore \int x^2 e^{3x}\,dx = x^2 \cdot \frac{e^{3x}}{3} - \int \frac{2x}{3} e^{3x}\,dx$$

$$= \frac{x^2 e^{3x}}{3} + \int \frac{-2x}{3} e^{3x}\,dx$$

Now set $u = \frac{-2x}{3}$ $\quad dv = e^{3x}\,dx$

$du = -\frac{2}{3}\,dx \quad v = \frac{e^{3x}}{3}$

Hence

$$\int -\frac{2x}{3} e^{3x}\,dx = -\frac{2x}{9} e^{3x} + \int \frac{2e^{3x}}{9}\,dx$$

$$= -\frac{2x}{9} e^{3x} + \frac{2e^{3x}}{27} + C$$

Together, we get:

$$\int x^2 e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \frac{2x}{9} e^{3x} + \frac{2e^{3x}}{27} + C$$

Ex. Compute $\int \sin^{-1}x \, dx$

Set $u = \sin^{-1}x \quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = x$

Hence $\int \sin^{-1}x \, dx = x \sin^{-1}x - \int \frac{x \, dx}{\sqrt{1-x^2}}$

To compute $\int \frac{-x \, dx}{\sqrt{1-x^2}}$,

set $u = 1 - x^2 \quad du = -2x \, dx$

$\rightarrow \quad -x \, dx = \frac{du}{2}$

$$\therefore \int \frac{-x\,dx}{\sqrt{1-x^2}} = \int \frac{\frac{du}{2}}{\sqrt{u}}$$

$$= \frac{1}{2} \cdot 2\sqrt{u} = \sqrt{1-x^2}$$

Together, we get

$$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

Ex. Compute $\int \tan^{-1}x\,dx$

Set $u = \tan^{-1}x \qquad dv = dx$

$du = \dfrac{dx}{1+x^2} \qquad v = x$

$$= x\tan^{-1}x - \left[\int \frac{\frac{du}{2}}{u}\right]$$

$$= x\tan^{-1}x - \frac{1}{2}\ln|u|$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

Ex. Compute $\int e^{2x}\sin 3x\,dx$

Set $u = e^{2x}$ $\quad dv = \sin 3x\,dx$

$du = 2e^{2x}dx \quad v = -\frac{\cos 3x}{3}$

Let $I = \int e^{2x} \sin 3x \, dx$.

We've shown:

$$I = -\frac{1}{3} e^{2x} \cos 3x$$

$$+ \frac{2}{9} e^{2x} \sin 3x \quad - \frac{4}{9} I$$

$$\therefore \quad I + \frac{4}{9} I = -\frac{1}{3} e^{2x} \cos 3x$$

$$\parallel \qquad\qquad + \frac{2}{9} e^{2x} \sin 3x$$

$$\frac{13}{9} I$$

Multiply by $\frac{9}{13}$

$\therefore \; I = \int e^{2x} \sin 3x \, dx$

$= \dfrac{-3}{13} e^{2x} \cos 3x + \dfrac{2}{13} e^{2x} \sin 3x + C$

Definite Integrals

~~Find~~ $\cancel{\int_0^{\pi/4}}$ Recall, if f and g are differentiable, then

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$$

Integrate from a to b:

$$\int_a^b (f(x)g(x))'\,dx$$

$$= \int_a^b f(x)g'(x)\,dx + \int_a^b g(x)f'(x)\,dx$$

Recall Part 2 of Fund. Thm. of Calc

says:

$$\int_a^b h'(x)\,dx = h(b) - h(a)$$

$$= h(x)\Big|_a^b$$

$$\therefore \int_a^b \left(f(x)g(x)\right)' dx = f(x)g(x)\Big|_a^b$$

Hence:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b$$

$$- \int_a^b g(x)f'(x)\,dx$$

Ex. Compute $\displaystyle\int_0^{1/2} \sin^{-1}x \, dx$

$u = \sin^{-1}x \qquad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}} \qquad v = x$

$$\int_0^{1/2} \sin^{-1}x \, dx = x\sin^{-1}x \Big|_0^{1/2} - \int_0^{\frac{1}{2}} \frac{x\,dx}{\sqrt{1-x^2}}$$

$u = 1-x^2$

$du = -2x\,dx$

$\dfrac{du}{2} = -x\,dx$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \int_1^{3/4} \frac{du}{2\sqrt{u}}$$

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$$= \frac{\pi}{12} + \frac{2}{2}\sqrt{u}\Big|_{1}^{3/4}$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

Reduction Formulas:

Compute $\int \sin^n x \, dx$

Set $u = \sin^{n-1} x \quad dv = \sin x \, dx$

$du = (n-1)\sin^{n-2}(\cos x)$

$v = -\cos x$

$$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \cos^2 x \, dx$$

Since $\cos^2 x = 1 - \sin^2 x$

$$\int \sin^n x \, dx = -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \left(1 - \sin^2 x\right) dx$$

$$\left(1 + (n-1)\right)\int \sin^n x \, dx = -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \, dx$$

For $\int \sin^2 x \, dx$, use $n = 2$:

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx$$

$$\therefore \frac{3}{4} \int \sin^2 x \, dx = -\frac{3}{8} \sin x \cos x + \frac{3x}{8}$$

$$\therefore \int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3x}{8}$$