

14.3 Partial Derivatives

In calculus of 1 variable,

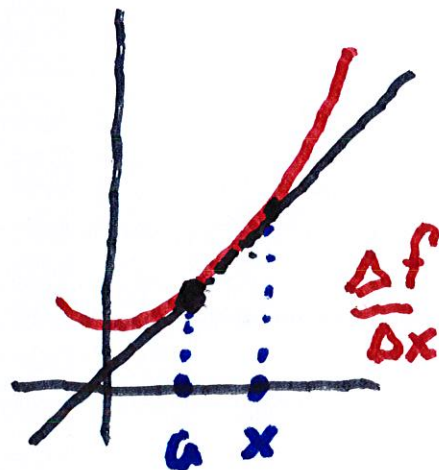
the derivative

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

tells how a function $f(x)$

is changing

when x is near a



Now suppose

$f(x, y)$ depends on 2 variables.

How does $f(x, y)$ change when we only allow x to change?

Thus we define

$$f'_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

And if we only allow y to change?

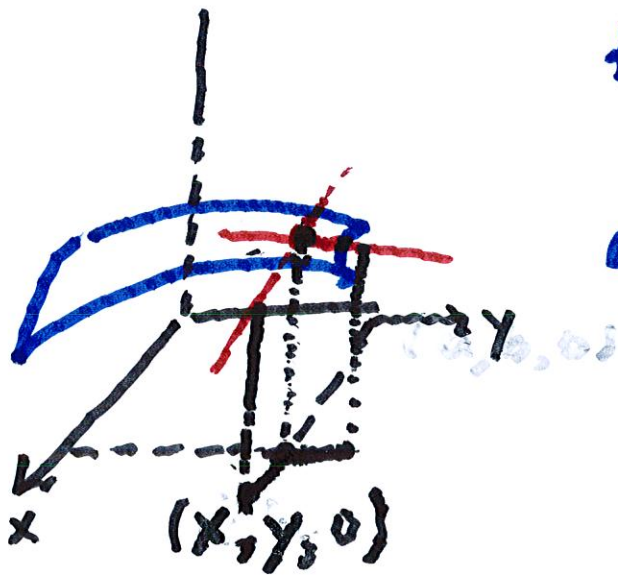
$$f'_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

If we now let the point (a, b)

to vary:

$$f'_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f'_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$f'_x(x, y)$ and $f'_y(x, y)$
are called
partial derivatives
of f at (x, y)

There are many alternative notations for

$$\begin{aligned}f_x(x, y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) \\ &= \frac{\partial z}{\partial x} = D_1 f = D_x f\end{aligned}$$

and

$$\begin{aligned}f_y(x, y) &= f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) \\ &= \frac{\partial z}{\partial y} = D_2 f = D_y f\end{aligned}$$

Rule for finding Partial Derivatives

$$\text{of } z = f(x, y)$$

1. To find f'_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f'_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

Ex. 1 If $f(x, y) = x^4 + x^2y^3 - 2y^3$

For f_x , keep y constant.

Find $f_x(1, 2)$

$$\begin{aligned} f_x(x, y) &= 4x^3 + 2xy^3 = 4 \cdot 1 + 2 \cdot 1 \cdot 8 \\ &= \underline{\underline{20}} \end{aligned}$$

For f_y , keep x constant.

Find $f_y(1, 2)$

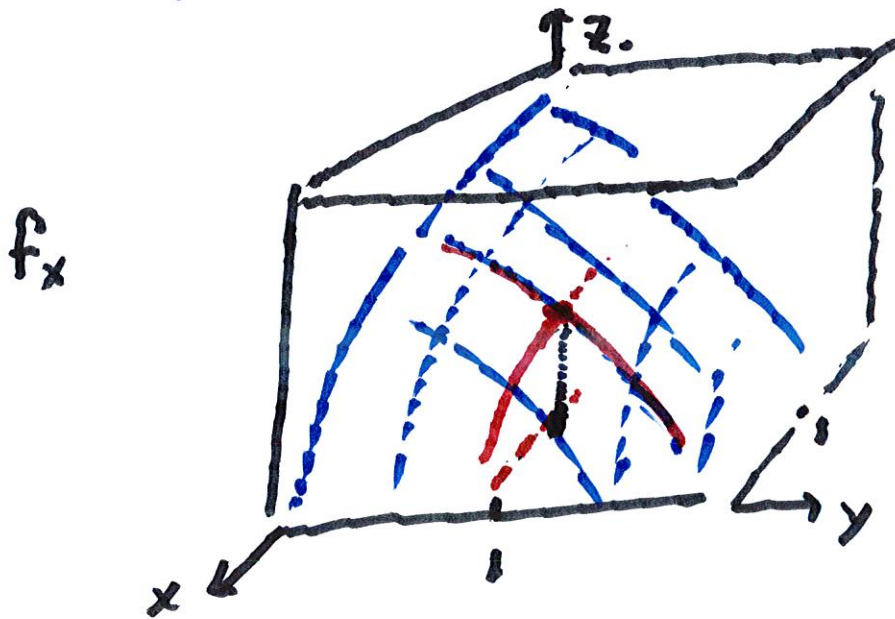
$$\begin{aligned} f_y(1, 2) &= 3x^2y^2 - 6y^2 = 3 \cdot 1 \cdot 4 - 6 \cdot 4 \\ &= \underline{\underline{-12}} \end{aligned}$$

Ex. If $f(x,y) = 4 - x^2 - 2y^2$,

find $f_x(1,1)$ and $f_y(1,1)$.

$$f_x = -2x = -2$$

$$f_y = -4y = -4$$



$$f_x(1,1) = -2 = \text{rate of change}$$

as x changes at $(1,1)$

Similarly,

$$f_y(1,1) = -4 = \text{rate of changing} \\ \text{at } (1,1)$$

We can use all of the rules
for differentiation to

compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\text{if } f(x, y) = \cos\left\{\frac{x^2}{1+y^3}\right\}$$

$$\frac{\partial f}{\partial x} = -\sin\left\{\frac{x^2}{1+y^3}\right\} \cdot \frac{2x}{1+y^3}$$

$$\frac{\partial f}{\partial y} = -\sin\left\{\frac{x^2}{1+y^3}\right\} \left(\frac{-x^2}{(1+y^3)^2}\right) 3y^2$$

Use implicit differentiation

to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

if z is defined implicitly by

$$x^2 + 2y^3 + z^4 - x^2y + z + 1 = 0$$

\downarrow
 $\hookrightarrow (z(x))^4$ $z(x)$

First with respect to x :

$$2x + 6y + 4z^3 \frac{\partial z}{\partial x} - 2xy + \frac{\partial z}{\partial x} = 0$$

(Always think of z as $z(x, y)$)

Solve for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} (1 + 4z^3) = 2xy - 2x$$

$$\therefore \frac{\partial z}{\partial x} = \frac{2xy - 2x}{1 + 4z^3} \quad \left(= \frac{6}{31} \right)$$

at (1, -2, -2)

Now with respect to y :

$$6y^2 + 4z^3 \frac{\partial z}{\partial y} - x^2 + \frac{\partial z}{\partial y} = 0$$

Solve for $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y} (1 + 4z^3) = x^2 - 6y^2$$

$$\rightarrow \frac{\partial z}{\partial y} = \frac{x^2 - 6y^2}{1 + 4z^3} \quad \left(= \frac{23}{31} \right) \\ \text{at } (1, -2, -2)$$

The problem is that we

don't know what z is.

When $x=1$, $y=-2$, then

$z = -2$ is a solution, i.e.,

$$z(1, -2) = -2$$

For functions of 3 variables:

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

To differentiate, keep y and z fixed.

Ex. If $f(x, y, z) = \frac{xy^2}{x+y+z}$,

find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \frac{(x+y+z)y^2 - xy^2 \cdot 1}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y+z)2xy - xy^2 \cdot 1}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial z} = \frac{-xy^2}{(x+y+z)^2}$$

Ex. Find the first derivatives
of $w = x^{y/z}$, if x, y, z
are > 0

13.1

Recall $\frac{d}{dx} (x^t) = t x^{t-1}$

$$\therefore \frac{\partial}{\partial x} (x^{y/z}) = \left(\frac{y}{z}\right) \cdot x^{y/z - 1}$$

$$\text{Also, } \frac{\partial}{\partial y} (x^{y/z}) = (\ln x) x^{y/z} \cdot \frac{1}{z}$$

→

$$\text{and } \frac{\partial}{\partial z} (x^{y/z}) = (\ln x) x^{y/z} \cdot \left(-\frac{x}{z^2}\right)$$

↑

$$\text{We used } \frac{d}{dt} (x^t) = (\ln x) x^t$$

Ex. Find first derivatives

$$\text{if } w = x^{(y/z)}$$

$$x = e^{\ln x}$$

$$\therefore x^{(y/z)} = \{e^{\ln x}\}^{y/z}$$

$$= e^{\frac{y \ln x}{z}}$$

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left\{ e^{\frac{y \ln x}{z}} \right\}$$

$$= e^{\frac{y \ln x}{z}} \cdot \frac{y}{xz}$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left(e^{\frac{y \ln x}{z}} \right)$$

$$= e^{\frac{y \ln x}{z}} \cdot \frac{\ln x}{z}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \left(e^{\frac{y \ln x}{z}} \right)$$

$$= e^{\frac{y \ln x}{z}} \cdot \left(-\frac{y \ln x}{z^2} \right)$$

Higher Derivatives

If $z = f(x, y)$, then

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

Similar formulas for $(f_x)_y$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

$$f_{12} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Ex. If $f(x, y) = x^3 + x^2y^4 + 3y^2$

$$f_x = 3x^2 + 2xy^4$$

$$(f_x)_y = 8xy^3$$

$$f_y (f_x)_x = 6x + 2y^4$$

$$f_y = 4x^2y^3 + 6y$$

$$(f_y)_x = 8xy^3$$

$$(f_y)_y = 12x^2y^2 + 6$$

Note that

$$(f_x)_y = 8xy^3 = (f_y)_x$$

Thm. If f_{xy} and f_{yx} are

continuous in a disk D , then

$$f_{xy} = f_{yx}$$

Ex. Compute f_{xy_2} if

$$f(x, y, z) = x^2y - y^2z^3x^2$$

$$f_x = 2xy - 2xy^2z^3$$

$$f_{xy} = 2x - 4xy^2z^3$$

$$f_{xyz} = -12xyz^2$$