14.4 continued.

0.1

Ex. Pressure, volume and temperature of a gas are related by $PV = 8.31\,T$, where P is measured in kilopascals V in liters, and T in kelvins.

Find the approximate change in pressure if the volume increases from 12 L to 12.3 L, and the temperature decreases from 310 K to 305 K

Use differentials

$PV = CT,$ where $C = 8.31$

$$P = \frac{CT}{V}$$

We use $V = 12$

and $T = 310$

$dV = .3$, $dT = -5$

$$dP = \frac{C}{V}\,dT - \frac{CT}{V^2}\,dV$$

$$= 8.31\left(\frac{-5}{12} - \frac{310}{12^2}(.3)\right)$$

$$= 8.31\left(\frac{-60 - 93}{12^2}\right) = -8.83$$

14.5 Chain Rule

If $y = f(x)$ and $x = g(t)$,

we can form the composition

$y(t) = f(g(t))$.

The Chain Rule says

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt},$$

or more preisely

$$\frac{d}{dt}\left(f(g(t)) \right) = \frac{df}{dt}(g(t))\,\frac{dg}{dt}(t)$$

Chain Rule (Case 1)

Suppose $z = f(x, y)$ is differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t.

Then $z = f(g(t), h(t))$ is a

differentiable function of t.

and $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Proof: $$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

and $\Delta x \approx \dfrac{dx}{dt}\Delta t$ and

$$\Delta y \approx \frac{dy}{dt}\Delta t$$

If we divide by Δt

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}.$$

Now let $\Delta t \to 0$. We get

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

or :

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Note that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x(t), y(t))$

and $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x(t), y(t))$

Ex. If $z = x^2 y + x y^3$,

where $x = e^{2t}$ and $y = 2e^{3t}$.

then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= (2xy + y^3)(2e^{2t})$

$+ (x^2 + 3xy^2)(6e^{3t})$

Ex. The radius of a cone increases at a rate of 1.5 inches/sec. Also the height decreases at 2 in/s.

At what rate does the volume change when $r = 10$ in. and $h = 8$ in?

$$V = \frac{\pi r^2 h}{3} \qquad \frac{dr}{dt} = 1.5 \quad \frac{dh}{dt} = -2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

Note that $\frac{\partial V}{\partial r} = \frac{2\pi r h}{3} = \frac{2\pi}{3}(80)$

and $\frac{\partial V}{\partial h} = \frac{\pi r^2}{3} = \frac{\pi}{3}(100)$

$$\frac{dV}{dt} = \frac{2\pi}{3}(80)(1.5) + \frac{\pi}{3}(100)(-2)$$

$$= \frac{40\pi}{3}$$

The Chain Rule (Case 2)

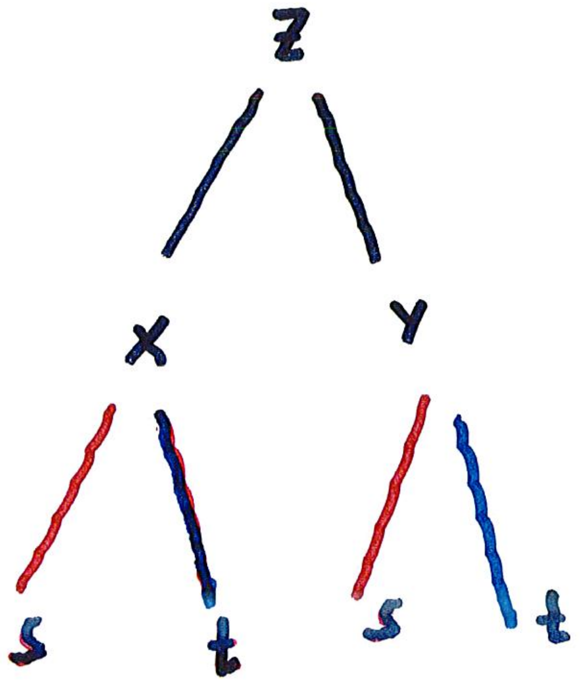
Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

It may be useful to look at the tree diagram:



s and t are the independent variables and Z is the dependent variable.

x and y are the intermediate variables.

Also, when we compute $\frac{\partial z}{\partial s}$ we fix t, and differentiate with respect to s. Hence we use Case 1 of the Chain Rule

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

and $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$$

Ex. Let $W(s,t) = F(u(s,t), v(s,t))$

where F, u, and v are

differentiable and satisfy

$U(1,0) = 2$ $\qquad$ $v(1,0) = -3$

$U_s(1,0) = -2$ $\qquad$ $v_s(1,0) = 5$

$U_t(1,0) = 6$ $\qquad$ $v_t(1,0) = 4$

$F_u(2,3) = -1$ $\qquad$ $F_v(2,3) = 10$

Compute $W_t(1,0)$

10.1

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$= \frac{\partial F}{\partial u}(2,3) \cdot \frac{\partial u}{\partial t}(1,0) + \frac{\partial F}{\partial v}(2,3) \cdot \frac{\partial v}{\partial t}(1,0)$$

$$= (-1)(6) + 10 \cdot 4 = 34$$

Similarly $\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial s}$

$$= \frac{\partial F}{\partial u}(2,3) \cdot \frac{\partial u}{\partial s} \quad \frac{\partial F}{\partial v}(2,3) \cdot \frac{\partial v}{\partial s}(1,0)$$

$$= (-1)(-2) + 10 \cdot 5 = 52$$

$$\therefore \frac{\partial W}{\partial s}(1,0) = 52$$

and $\frac{\partial W}{\partial t}(1,0) = 34$

The Chain Rule works for any number of variables:

Ex. Suppose $u = x^4 y + y^2 z^3$,

where

$x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial u}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)r^2 \sin t$$

To compute this for $r = 2$, $s = 1$ and $t = 0$, we have to compute the values of x, y, and z

We get $x = 2$, $y = 2$, and $z = 0$