14.6 Directional Derivatives

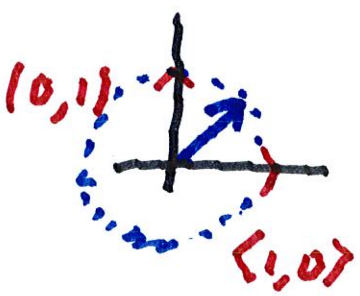
$$\lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \text{rate of change}$$

in the $\langle 1,0 \rangle$ direction

and

$$\lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = \text{rate of change in the}$$

$\langle 0,1 \rangle$ direction



Note that $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$ are unit vectors.

Now let $\vec{u}$ be any unit vector

We define

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Ex. Suppose $x = x_0 + ha$ and $y = y_0 + hb$

We want to figure out the rate of change as $h \to 0$ when $h = 0$

We define $g(h) = f(x_0 + ah, y_0 + bh)$

where $\langle a, b \rangle$ is a unit vector.

By the Chain Rule, we get

$$\frac{dg}{\partial t} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

OR $$\frac{dg}{\partial t} = \frac{\partial f}{\partial x}(x_0, y_0)\, a + \frac{\partial g}{\partial y}(x_0, y_0)\, b$$

$$\frac{df}{dh} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh}$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0)\, b$$

or:

$$D_{\vec{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

This is called the directional derivative of f at (x, y) in the direction of a unit vector $\vec{v}$

Ex. Let $f(x,y) = x^2 y - y^2 - x^3$

Compute $D_{\vec{u}} f(x_0, y_0)$

at $x_0 = 1,\ y_0 = 2$ and $\vec{u} = \frac{3}{\sqrt{10}} \vec{i} + \frac{1}{\sqrt{10}} \vec{j}$

$$\frac{\partial f}{\partial x} = 2xy - 3x^2 = 4 - 3 = 1$$

$$\frac{\partial f}{\partial y} = x^2 - 2y = -3$$

$$D_{\vec{u}} f = 1 \cdot \frac{3}{\sqrt{10}} - 3 \cdot \frac{1}{\sqrt{10}} = 0$$

If we use [illegible]

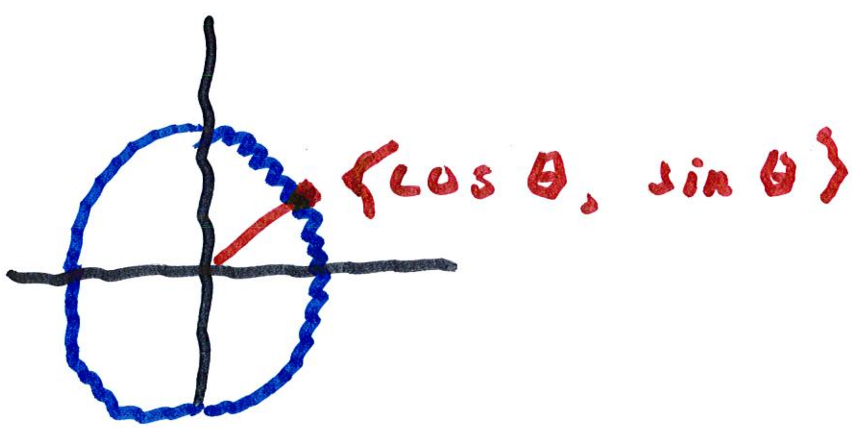
$\vec{v}$ that makes an angle of θ

with the x-axis, then

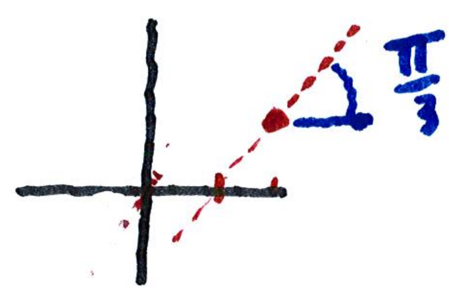
$$\frac{dg}{dt} = \frac{\partial f}{\partial x}(x_0, y_0)\cos\theta + \frac{\partial f}{\partial y}(x_0, y_0)\sin\theta$$

is the direction of

$\vec{u} = \langle \cos\theta, \sin\theta \rangle$.



Ex. Find the directional derivative $D_{\vec{u}} f(x,y)$ of $f(x,y) = x^2 - xy^3 + y^2$ and $\vec{u}$ is the unit vector pointing in the direction with angle $\theta = \frac{\pi}{3}$ and $(x_0, y_0) = (2,1)$.



$$f_x = 2x - y^3 = 4 - 1 = 3$$

and

$$f_y = -3xy^2 + 2y = -6 + 2 = -4$$

Since $\vec{v} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$,

$$D_{\vec{v}} f(2,1) = 3 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2} - 2\sqrt{3}$$

Note that we can write

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

We call this the gradient of f at (x,y).

$$\nabla f_{(x,y)} = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

Hence $D_{\vec{u}} f(x,y) = \underline{\underline{\nabla f(x,y) \cdot \vec{u}}}$

Ex. Find the directional derivative

of the function $f(x,y) = x^2y^3 - y^2$

at the point $(3, 2)$ in the

direction of $\vec{u} = \dfrac{2\vec{i} + 3\vec{j}}{\sqrt{13}}$

$$\frac{\partial f}{\partial x} = 2xy^3 = 6 \cdot 8 = 48$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 2y = 108 - 4 = 104$$

$$\therefore \nabla f_{(3,2)} = 48\vec{i} + 104\vec{j}$$

$$\Rightarrow D_{\vec{u}} f = 48 \cdot \frac{2}{\sqrt{13}} + \frac{104 \cdot 3}{\sqrt{13}}$$

$$= \frac{408}{\sqrt{13}}$$

Functions of 3 variables:

Given a function $f(x, y, z)$

and a unit vector $\vec{u} = \langle a, b, c \rangle$

at (x_0, y_0, z_0), then we define

$$D f_{\vec{u}}(x_0, y_0, z_0)$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

We can compute $D_{\vec{u}} f(x, y, z)$

$$= \frac{\partial f}{\partial x}(x, y, z)\, a + \frac{\partial f}{\partial y}(x, y, z)\, b + \frac{\partial f}{\partial z}(x, y, z)\, c$$

We can define the directional derivative by $\nabla f =$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$$

$D_{\vec{v}} f = \nabla f \cdot \vec{v}$

The quantity $\nabla f \cdot \vec{v}$

depends on which vector we choose.

Recall the formula $\vec{a} \cdot \vec{b} = |\vec{a}||b| \cos\theta$

$$\nabla f \cdot \vec{v} = |\nabla f||\vec{v}| \cos\theta$$

$$= |\nabla f| \cos\theta$$

We get the largest value when

$\cos\theta = 1$, i.e., when $\vec{u}$ points

in the same direction as ∇f.

But remember, $\vec{u}$ must be a

unit vector. i.e, when

$$\vec{u} = \frac{\nabla f}{|\nabla f|}$$

Then $\nabla f \cdot \vec{u}$

$$= \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|}$$

$$= |\nabla f|$$

To minimize

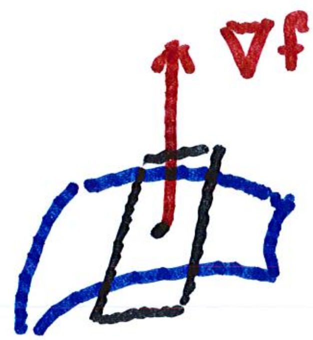
set $\vec{u} = -\dfrac{\nabla f}{|f|}$.

In sum, $\nabla f \cdot \vec{u}$ is maximized

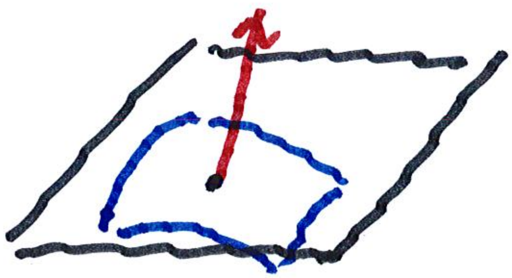
when $\vec{v} = \dfrac{\nabla f}{|\nabla f|}$ and the largest

value is $= |\nabla f|$

Tangent Planes to level surfaces.



Note that if we travel in the unit direction $\vec{u}$, then the rate of change is $= 0$ if the motion vector $\vec{u}$ satisfies $\nabla f \cdot \vec{u} = 0$.



We can define the tangent plane by

$$F_x(x_0, y_0, z_0)(x - x_0) \qquad (1)$$

$$+ F_y(x_0, y_0, z_0)(y - y_0)$$

$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Ex. Find the equation of the tangent plane of the level surface $f(x,y,z)=k$ through (x_0, y_0, z_0) by using (1).

It satisfies

$$F_x(x_0, y_0, z_0)(x-x_0)$$

$$+ F_y(x_0, y_0, z_0)(y-y_0)$$

$$+ F_z(x_0, y_0, z_0)(z-z_0) = 0.$$

Ex. Let S = surface defined

by $x^3 - xy + z^2 = 6$

Find the equation of the

tangent plane containing

$(1, -1, 2)$.

$F_x = 3x^2 - y + 2z$

$= 3 + 1 + 4 = 8$

$F_y = -x = -1$

$F_z = 2z = 4$

$\therefore$ Plane is

$$8(x-1) - (y+1) + 4(z-2) = 0$$

A surface $z = z(x, y)$ is

defined by

$$0 = F(x, y, z(x, y)) \quad \text{at } (x_0, y_0, z_0)$$

Compute $\dfrac{\partial z}{\partial x}(x, y)$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}(x, y)\frac{\partial z}{\partial x} = 0$$

or: $$\frac{\partial z}{\partial x} = \frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}$$

Ex. If $z(x,y)$ satisfies

$$x^2y - yz^2 + z^3 = 1,$$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial}{\partial x}\left(x^2y - yz^2 + z^3\right) = 0$$

$$-2xy - 2yz\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$$

$$\rightarrow \quad \frac{\partial z}{\partial x} = \frac{2xy}{3z^2 - 2yz}$$

$$\frac{\partial}{\partial y}\left(x^2 y - y z^2 + z^3 \right) = 0$$

$$x^2 - z^2 - 2yz \frac{\partial z}{\partial y} + 3z^2 \frac{\partial z}{\partial y} = 0$$

$$\rightarrow \quad \frac{\partial z}{\partial y} = \frac{z^2 - x^2}{3z^2 - 2yz}$$