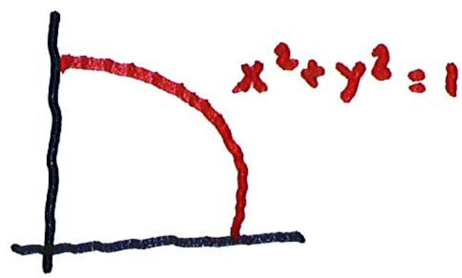
Ex. Find $(\bar{x}, \bar{y})$ if D = region in first ~~octant~~ quadrant bounded by $x^2+y^2=1$ and where the density $= y$



$$\bar{x} = \frac{M_y}{m}$$

$$m = \iint_D y \, dA = \int_0^{\pi/2} \int_0^1 r\sin\theta \; r\,dr\,d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= -\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = \frac{1}{3}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\theta \, r\sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta \cos\theta}{4} \, d\theta$$

$$= \frac{1}{4} \frac{\sin^2\theta}{2} \Big|_0^{\frac{\pi}{2}} = \frac{1}{8}$$

$$\therefore \hat{x} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}$$

$$M_x = \iint_D y \cdot y \, dA$$

$$= \int_0^{\pi/2} \int_0^1 (r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta$$

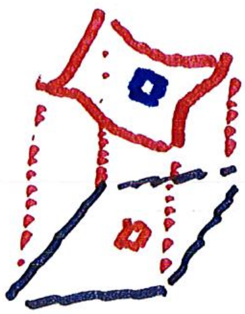
$$= \frac{\theta}{8} - \frac{\sin 2\theta}{16} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{16}$$

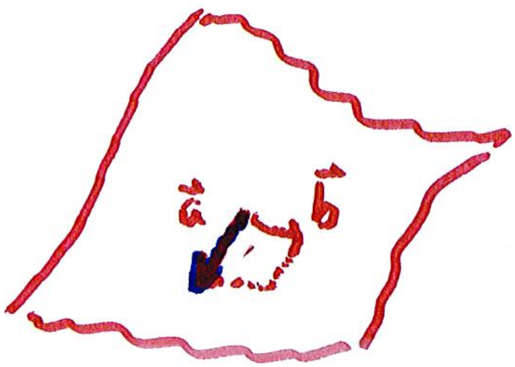
$$\therefore \bar{y} = \frac{M_x}{m} = \frac{\pi}{16} \cdot 3 = \frac{3\pi}{16}$$

15.5 B 14

~~15.6~~ Surface Area



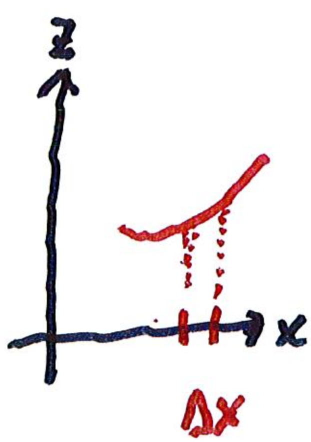
The surface $z = f(x, y)$



$\vec{a} = \Delta x \, \vec{i} + f_x \Delta x \, \vec{k}$ ✓

$\vec{b} = \Delta y \, \vec{j} + f_y \Delta y \, \vec{k}$

$= (\Delta x, \frac{\partial f}{\partial x} \Delta x)$ ✓



$(\Delta x, f'(x) \Delta x) = \vec{a}$

$(\Delta y, f'(x, y) \Delta y$

Compute area of parallelogram

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= -f_x \Delta x \Delta y \vec{i} - f_y \Delta x \Delta y \vec{j} + \Delta x \Delta y \vec{k}$$

$$|\text{Area}| = \left((f_x)^2 + (f_y)^2 + 1\right)^{\frac{1}{2}} \Delta x \Delta y$$

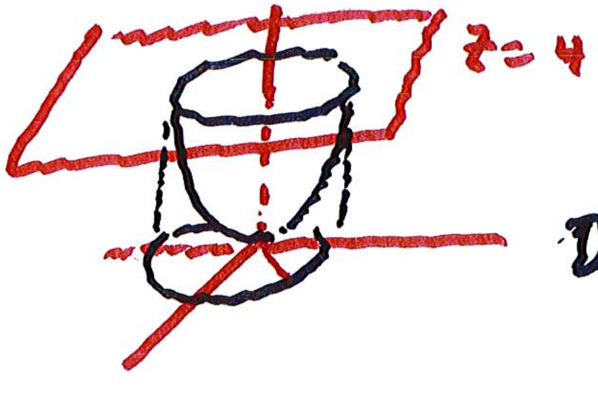
$\therefore$ Area of S above D is

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Ex. Find surface area of

paraboloid $z = x^2 + y^2$ below $z = 4$

$x^2 + y^2 = z = 4 \rightarrow r = 2$



$D = D_2(0)$

$z = x^2 + y^2 = f(x,y)$

$f_x = 2x \quad f_y = 2y$ $\qquad \iint\limits_D \sqrt{1+4x^2+4y^2}\, dA$

$\rightarrow \sqrt{1+4x^2+4y^2}$

$SA = \iint\limits_D \sqrt{1+4(x^2+y^2)}\, dA$

$$= \int_0^{2\pi} \int_0^{2} \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$= \frac{2\pi}{8} \int_0^2 \sqrt{1+4r^2}\; 8r\,dr$$

$u = 1 + 4r^2$

$du = 8r\,dr$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{u}\; du$$

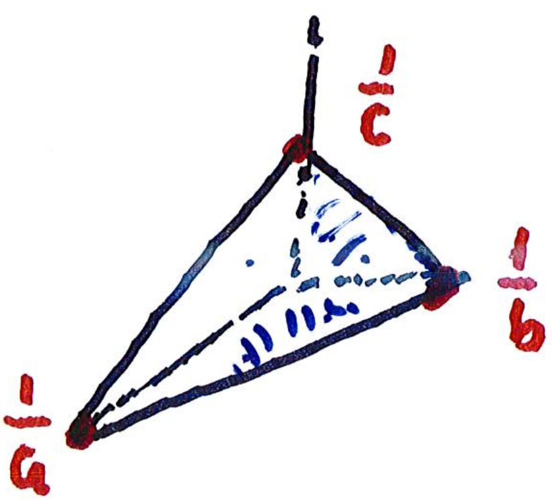
$$= \frac{\pi}{4} \cdot \frac{2}{3} (u)^{3/2} \Big|_1^{17}$$

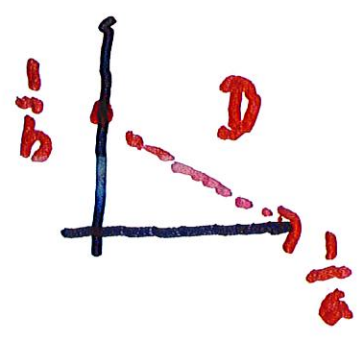
$$= \frac{\pi}{6} \left(17^{3/2} - 1 \right)$$

Ex. Find surface area of

$ax + by + cz = 1$ above

the xy-plane in 1st octant





$$z = \frac{1 - ax - by}{c}$$

$$\frac{\partial f}{\partial x} = -\frac{a}{c} \qquad \frac{\partial f}{\partial y} = -\frac{b}{c}$$

$$A = \iint_D \sqrt{1 + \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2}\, dx\,dy$$

$$= \frac{\sqrt{c^2 + a^2 + b^2}}{c} \iint_D 1 \cdot dA$$

$$\downarrow$$

$$= \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2+b^2+c^2}}{c} \cdot \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2+b^2+c^2}}{2abc}$$

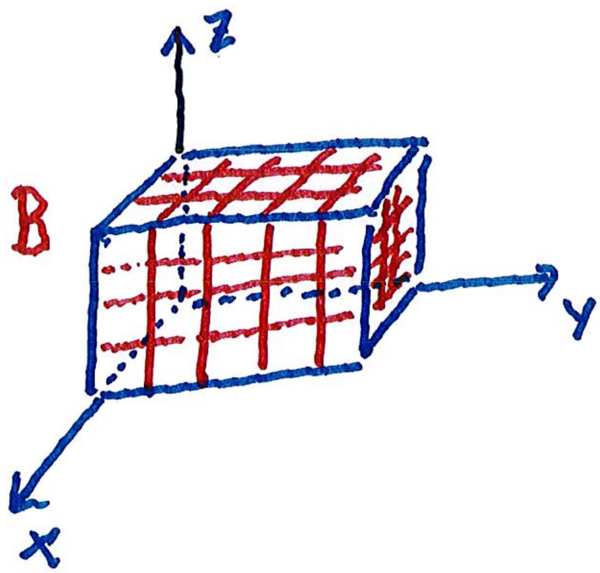
~~15.7~~ 15.6 Triple Integrals A

Given a box $B = \left\{ (x,y,z) \,\middle|\, \begin{matrix} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{matrix} \right\}$,

we divide B into sub-boxes

of width Δx, Δy, and Δz,

where $\Delta x = \dfrac{b-a}{\ell}$, $\Delta y = \dfrac{d-c}{m}$, $\Delta z = \dfrac{s-r}{n}$



We define the triple Riemann sum by

$$\sum_{i=1}\sum_{j=1}\sum_{k=1} f(x_i, y_j, z_k)\,\Delta x\,\Delta y\,\Delta z,$$

where (x_i, y_j, z_k) is in the box

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

If we let $\ell, m, n \to \infty$

$$\iiint_B f(x,y,z)\,dV = \text{limit of Riemann sum as } \ell, m, n \to \infty$$

This is called the triple integral of f over the box B. To calculate it, we use:

Fubini's Thm.

$$\iiint_B f(x,y,z) = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Integrate first with respect to x, then y, then z.

Or one could integrate first with respect to y, then z, then x, etc.

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\, dy\, dz\, dx, \text{ etc.}$$

Ex. If $B = \left\{ (x,y,z) \,\middle|\, 0 \le x \le 2,\ 1 \le y \le 2,\ 0 \le z \le 1 \right\}$

calculate $\displaystyle\iiint_B xy^2 z \; dz\, dy\, dx$

$$= \int_0^2 \int_1^2 \int_0^1 xy^2z \, dz \, dy \, dx$$

$$= \int_0^2 \int_1^2 \frac{xy^2z^2}{2} \Big|_0^1 \, dy \, dx$$

$$= \int_0^2 \int_1^2 \frac{xy^2}{2} \, dy \, dx$$

$$= \int_0^2 \frac{xy^3}{6} \Big|_{y=1}^{y=2} \, dx$$

$$= \int_0^2 \left(\frac{x \cdot 4}{3} - \frac{x}{6} \right) dx = \int_0^2 \frac{7x}{6} \, dx$$

$$\frac{7x^3}{12}\Big|_0^2 = \frac{7}{3}$$

A solid region E is said to be of type 1, if it lies between the graphs of 2 functions of x and y over a region D:

$$\therefore E = \left\{ (x, y, z) \mid (x, y) \in D, \ u_1(x, y) \leq z \leq u_2(x, y) \right\}$$

Type 1

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

If D is a type II region, then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$$

dx dy

or if D is of type I:

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)$$

dz dy dx

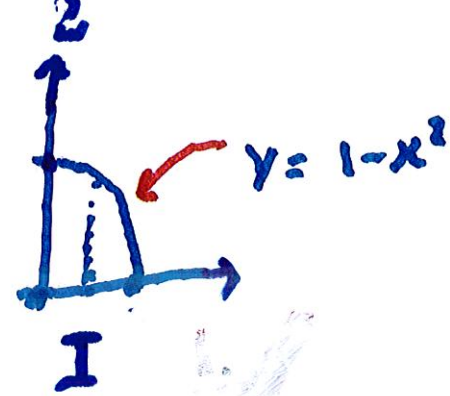
Ex. Let E = solid region bounded

by $z = x^2 + 2y + 1$ and $z = y + 2$

in the first octant. Find Vol.

$$x^2 + 2y + 1 = y + 2$$

$$\rightarrow \quad y = 1 - x^2$$



$$\text{Vol} = \iiint 1 \, dV$$

$$= \int_0^1 \int_0^{1-x^2} \int_{x^2+2y+1}^{y+2} 1 \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} z \Big|_{x^2+2y+1}^{y+2} \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} (y+2) - (x^2+2y+1) \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} (1 - x^2 - y) \, dy\, dx$$

$$= \cancel{\int_0^1 \Big[y - x^2 y - \frac{y^2}{2} \Big]^{1-x^2}}$$

$$= \int_0^1 y(1-x^2) - \frac{y^2}{2} \Big|_0^{1-x^2} \, dx$$

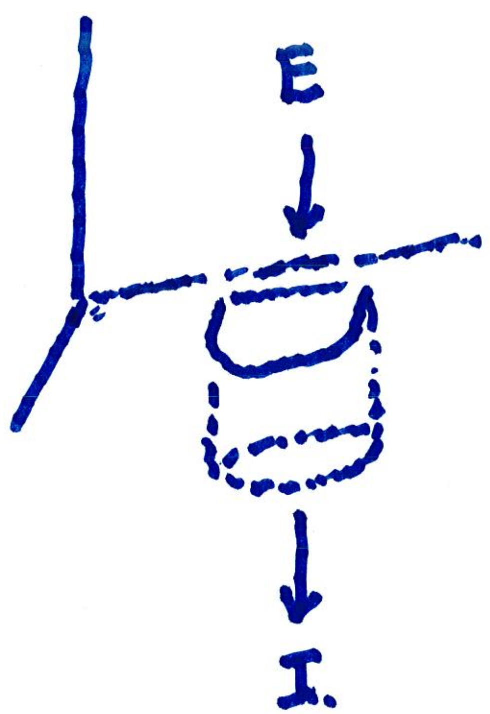
Remarks

1. At first glance, it might seem better to treat every region E as a Type I. region

Ex. $\iint_D (x^2+y^2)^2 \, dA$, 

where $\begin{Bmatrix} 1 < x^2+y^2 < 4, \\ y>x, \quad y>-x \end{Bmatrix}$ is MUCH EASIER to compute by using polar coordinates

2. One major difficulty is to find an interval $[a, b] = I$ and a domain D, so that D lies on I and E lies above D.



$z = h_2(x)$

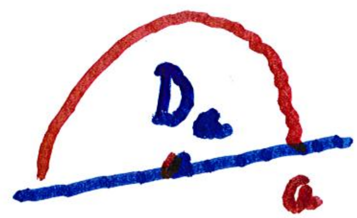
$z = h_1(x)$

$y = g_2(x)$ $\quad y = g_1(x)$.

Ex. Let $D_a = \left\{ (x,y) \,\middle|\, \begin{matrix} x^2+y^2 < a^2 \\ y > 0 \end{matrix} \right\}$

Also, assume that the density $= y$. Then, find $(\bar{x}, \bar{y})$.

$$M_x = \iint_{D_a} y \cdot y \, dA$$



$$= \int_0^{\pi} \int_0^{a} (r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \int_0^{\pi} \sin^2\theta \, d\theta \cdot \int_0^{a} r^3 \, dr$$

$$= \int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta \cdot \int_0^{a} r^3\, dr$$

$$= \frac{\pi}{2} \cdot \frac{a^4}{4} = \frac{\pi a^4}{8}$$

For m we have

$$m = \iint_{D_a} y\, dA = \int_0^{\pi} \int_0^{a} \underbrace{r\sin\theta}\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{a} r\sin\theta \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta\, d\theta \cdot \int_0^{a} r^2\, dr$$

$$= \left(-\cos\theta \Big|_0^{\pi}\right) \cdot \frac{a^3}{3}$$

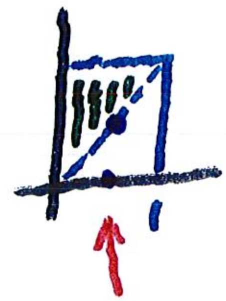
$$= \frac{2a^3}{3}.$$

Hence $\bar{y} = \dfrac{M_x}{m} = \dfrac{\frac{\pi a^4}{8}}{\frac{2a^3}{3}} = \boxed{\dfrac{3\pi a}{16}}$

By symmety about the y-axis,

$\bar{x} = 0. \Rightarrow (\bar{x}, \bar{y}) = \left(0, \dfrac{3\pi a}{16}\right)$

Ex. Let R = triangular region in the xy-plane between $y = x$ and $y = 1$ for $0 \le x \le 1$,



and let E = solid region between the surfaces $z = -2$ and $z = 1 - x^2$ for $(x, y) \in R$

Evaluate $\displaystyle\iint_R \int_{-2}^{1-x^2} (x+1)\, dz\, dy\, dx$

Evaluate $\displaystyle\iiint_E (x+1)\, dV$.

$$= \int_0^1 \int_x^1 \int_{-2}^{1-x^2} (x+1)\, dz\, dy\, dx$$

$$\int_0^1 \int_x^1 (x+1) z \Big|_{-2}^{1-x^2} dy\, dx$$

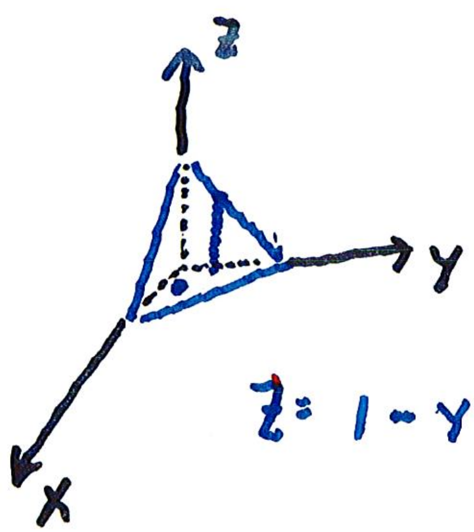
$$= \int_0^1 \int_x^1 (x+1)\left(1-x^2 - (-2) \right) dy\, dx$$

$$= \int_0^1 \int_x^1 (-x^3 - x^2 + 3x + 3)\, dy\, dx$$

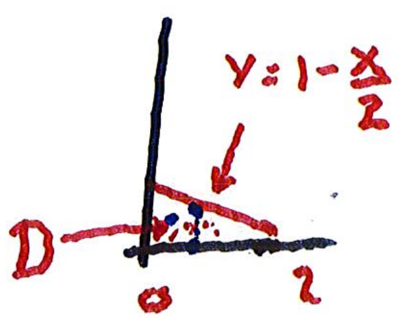
Ex. Suppose that a tetrahedron is bounded by $x + 2y + 2z = 2$, $x = 0$, $y = 0$ and $z = 0$.

and that the density is $\rho = 2z$. Calculate mass m

Set $z = 0 \rightarrow x + 2y = 2 \rightarrow y = 1 - \frac{x}{2}$



$z = 1 - y - \frac{x}{2}$)



$$= \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{1-y-\frac{x}{2}} 2z \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} \left(y + \frac{x}{2} - 1 \right)^2 dy \, dx$$

$$= \int_0^2 \left. \frac{\left(y + \frac{x}{2} - 1 \right)^3}{3} \right|_{y=0}^{y=1-\frac{x}{2}} dx$$

$$= \int_0^2 \frac{\left(1 - x/2 + \frac{x}{2} - 1\right)^3}{3} - \frac{\left(\frac{x}{2} - 1\right)^3}{3} \, dx$$

$$= \frac{1}{3}\int_0^2 \left(1 - \frac{x}{2}\right)^3 dx$$

$$= \frac{1}{3}\int_0^2 1 - \frac{3x}{2} + 3\frac{x^2}{4} - \frac{x^3}{8} \, dx$$

$$= \frac{1}{3}\left(x - \frac{3x^2}{4} + \frac{x^3}{4} - \frac{x^4}{32}\right)\Big|_0^2$$

$$= \frac{2 - 3 + 2 - \frac{1}{2}}{3} = \underline{\underline{\frac{1}{6}}}$$