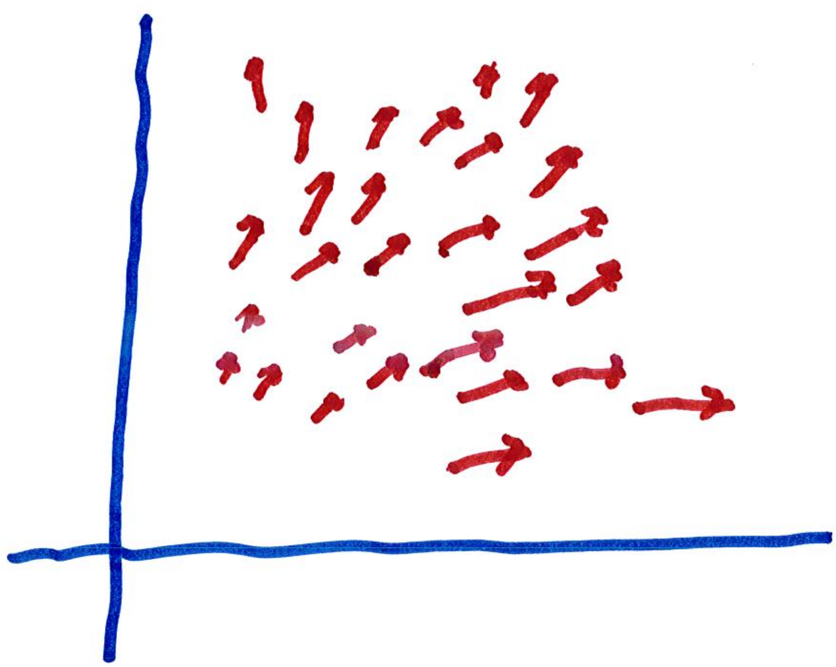
16.1 Vector Fields

Imagine a map showing the direction of the wind



At each point, there is an arrow. Also, if the wind is strong, then the arrow is bigger.

This is a vector field.

Def'n. Let D be a region in $\mathbb{R}^2$.

A vector field $\vec{F}$ is a function that assigns to each point (x,y) in D a vector (2-dimensional) $\vec{F}(x,y)$.

More precisely, we can write

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

Def'n. Let E be region in $\mathbb{R}^3$.

A vector field $\vec{F}$ on $\mathbb{R}^3$ is a function that assigns to each point (x, y, z) a 3-dimensional vector $\vec{F}(x,y,z)$

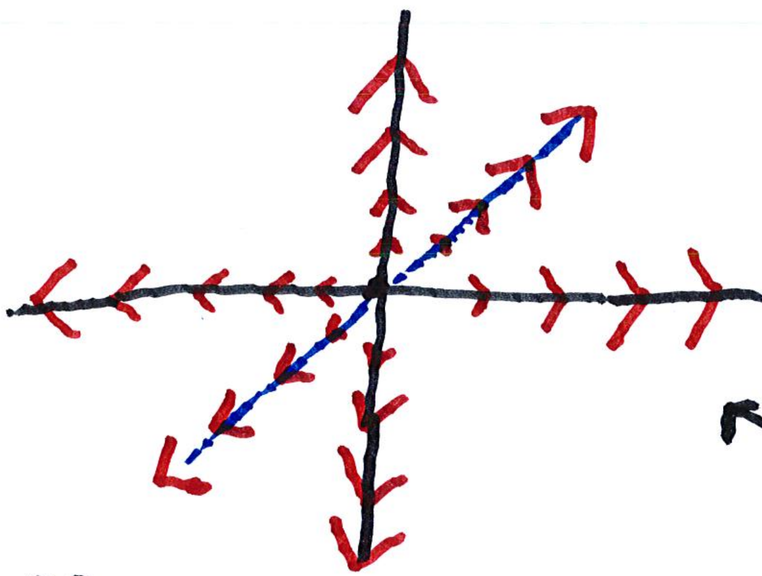
We can write $\vec{F}(x,y,z)$ as

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}.$$

We will say $\vec{F}$ is continuous on E if P, Q, and R are continuous functions of (x,y,z).

Ex. Sketch $\vec{F}(x,y) = \langle x, y \rangle$

$= x\vec{i} + y\vec{j}$



$F(x,x) = \langle x,x \rangle$

$F(x,0) = \langle x,0 \rangle$

if $y=0$

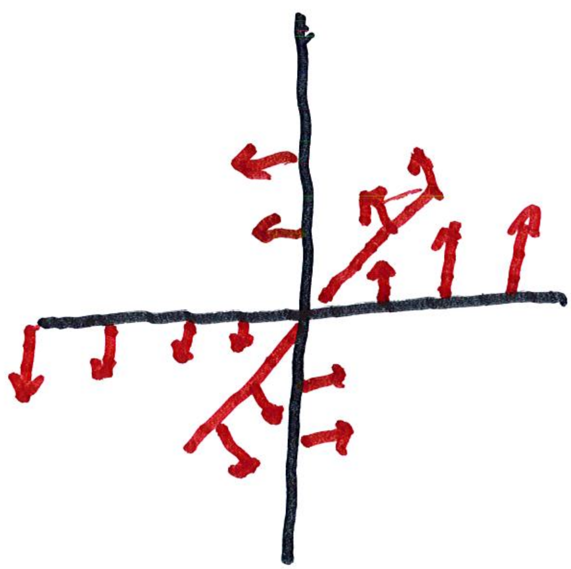
If $x=0$

$F(0,y) = \langle 0,y \rangle$

$\vec{F}$ points radially outward.

Rotational vector field

(like a hurricane)



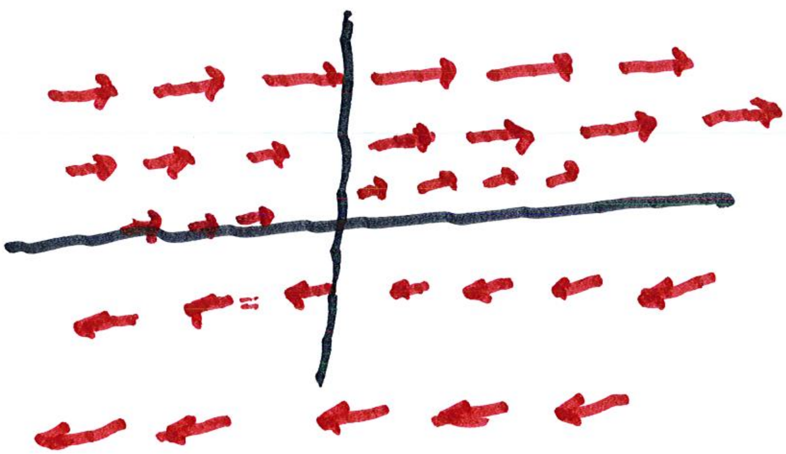
$\vec{F}(x,y) = \langle -y, x \rangle$

When $y=0$ $\vec{F} = \langle 0, x \rangle$

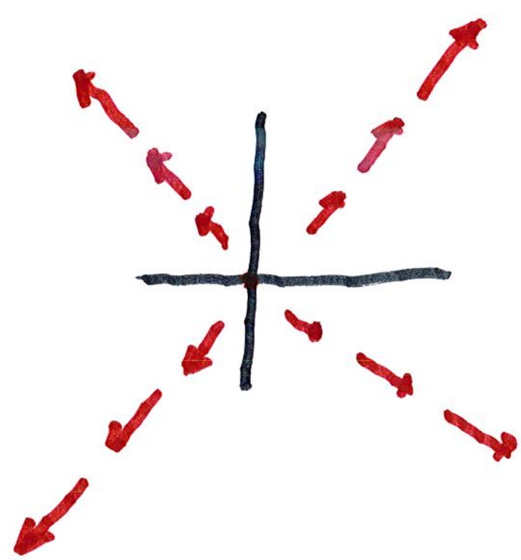
When $x=0$, $\vec{F} = \langle -y, 0 \rangle$

When $y=x$, $\vec{F} = \langle -x, x \rangle$

Doldroms $\vec{F}(x,y) = \langle y, 0 \rangle$



Gravity



1. $\vec{F}(x,y) = \langle x, y \rangle$

$|\langle x,y \rangle| = \sqrt{x^2+y^2}$

2. But gravity gets stronger as $(x,y) \to (0,0)$

$\vec{x} = \langle x, y \rangle$

$\vec{F} = \dfrac{\vec{x}}{|\vec{x}|}$ outward, always

has magnitude = 1

$$\vec{F} = \frac{\langle x, y \rangle}{|\langle x, y \rangle|} = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

We want the magnitude to be

inversely proportional to

the square of distance from (0,0)

$$\vec{F} = \left\langle \frac{x}{(x^2+y^2)^{3/2}} , \frac{y}{(x^2+y^2)^{3/2}} \right\rangle$$

But we want $\vec{F}$ to point inward

$$\vec{F} = \left\langle \frac{-x}{(x^2+y^2)^{3/2}} , \frac{-y}{(x^2+y^2)^{3/2}} \right\rangle$$

Finally we have to multiply

by the right constant C.

The U.S. economy or the world economy, etc.

Think of many possible quantities

1. Price of Steel
2. Price of Oil
3. Interest Rate
4. Price of wheat, etc.

One can model the economy

(based on many measures)

(say 100)

as a vector field in 100

. Given

$P_1, P_2, \ldots, P_{100}$, the vector

field measures ~~[illegible]~~ the

expected value of how $P_1, \ldots, P_{100}$

should change

i.e. $P_1' = a_{11} P_1 + \cdots + a_{1N} P_N + g_1$

$P_2' = a_{21} P_1 + \cdots + a_{2N} P_N + g_2$

$\vdots$

$P_N' = a_{N1} P_1 + \cdots + a_{NN} P_N + g_N$

If we set

$$\vec{P}(t) = \begin{pmatrix} P_1(t) \\ \vdots \\ P_{100}(t) \end{pmatrix}$$

and

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ & & \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \qquad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_N(t) \end{pmatrix}$$

then the N equations

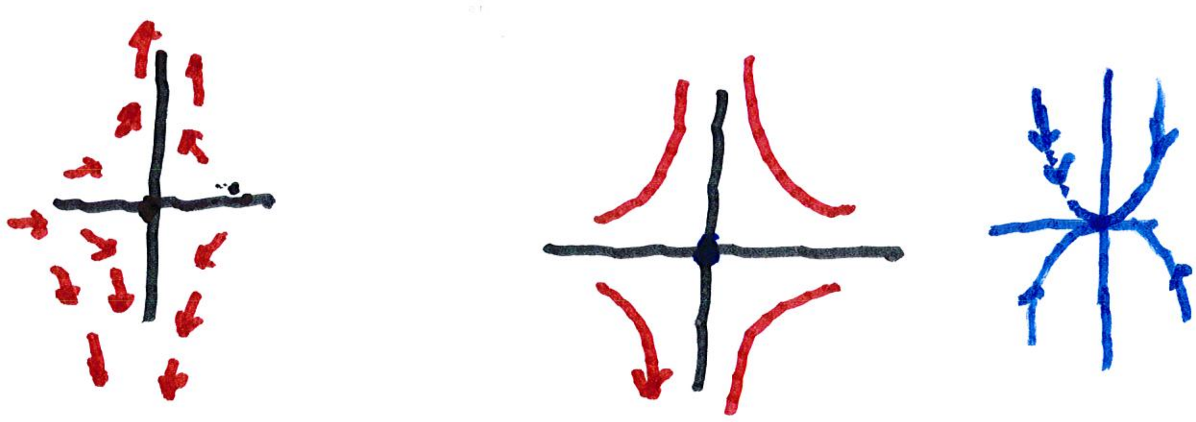
become

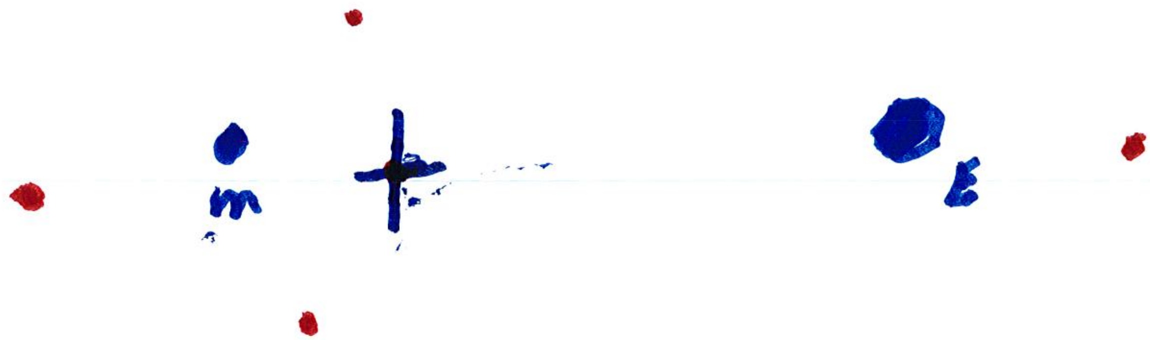
$$\vec{p}\,'(t) = A\,\vec{p}(t) + \vec{g}(t).$$

$\vec{g}(t)$ = external

force.

Five Lagrangean Points





Five Lag. Points.

Think of a building. The location of the joints gives quantities $p_1, \dots, p_N$

(Also, the velocity of the joints

gives quantities $Q_1, \ldots, Q_N$)

By Newton's 2nd (or 3rd) law

$$P_1'' = a_1^1 P_1 + \cdots + a_N^1 P_N + g_1(t)$$

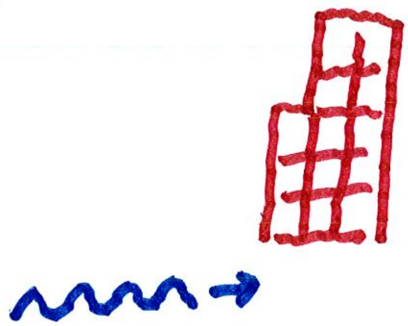
$$\vdots$$

$$P_N'' = a_1^N P_1 + \cdots + a_N^N P_N + g_N(t)$$

Say an earthquake happens

This is an external force

with a vibration.



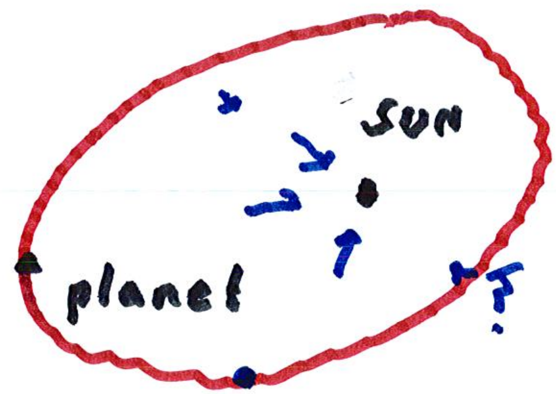
Does the frequency of the

earthquake match up

with the "natural frequency"

of the building (Resonance)

Galloping Gertie.



Kepler's Laws.

A planet travels about the sun, so that Sun is at the focus of an ellipse.

Newton invented calculus to show this.

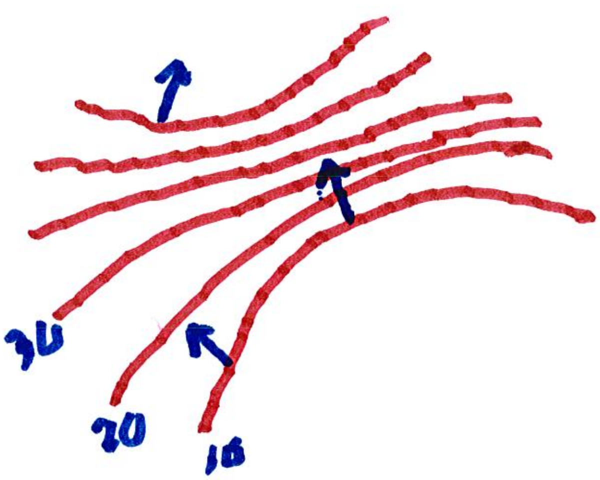
Given a function $f(x,y)$.

the gradient of f is

$$\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

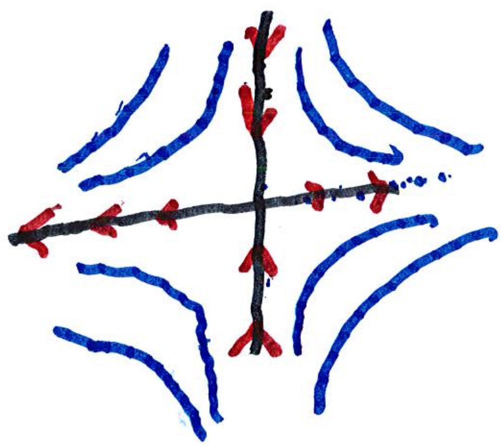
The gradient ∇f is always

$\perp$ to the level sets (level surfaces)



Sketch the ~~curve~~ vector field

$\vec{F} = x\vec{i} - y\vec{j}$



When $y=0$

$\langle x, 0 \rangle$

When

$x=0$ $\langle 0, -y \rangle$

16.2 Line Integrals

Suppose a curve C in the plane is described by

$(x(t), y(t))$ for $a \le t \le b$

We assume C is smooth, i.e., $\vec{r}\,'(t) \neq 0$ for all $t \in [a, b]$.

We want to define integral of a function $f(x,y)$ defined on C.

We subdivide $[a, b]$ in the usual way:

$a = t_0 < t_1 < \ldots < t_n = b$, where

$\Delta t = (t_i - t_{i-1}) = \frac{b-a}{n}$.

We let $P(t_i^*)$ be

any point on the curve C

with $P(t_i^*)$ on the curve

with $t_{i-1} \le t_i^* \le t_i$.

Then we define

$$\Delta s_i = \left| \langle x(t_i), y(t_i) \rangle - \langle x(t_{i-1}), y(t_{i-1}) \rangle \right|$$

This is a short vector

connecting the above two points.

Then we define $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$

If $f(x,y)$ is [illegible] on C,

then we can define

$$\int_C f(x,y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

The length Δs_i

$$ds \approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ \Delta t. \qquad (1)$$

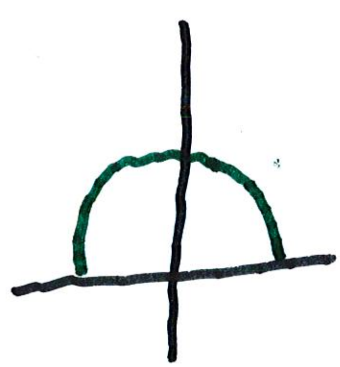
Ex. Evaluate $\displaystyle\int_C \left(3 + 2x^2 y\right) ds$

where C is the upper half

circle defined by

$x = \cos t$ and $y = \sin t$,

with $0 \le t \le \pi$



Formula (1) implies

$$\int_C \left(3 + 2\cos^2 t \cdot \sin t\right) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \left(3 + 2\cos^2 t \cdot \sin t\right) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi} \left(3 + 2\cos^2 t \sin t\right) \cdot 1 \cdot dt$$

$$= \left[3t - \frac{2}{3}\cos^3 t \right]_0^{\pi}$$

$$= \left(3\pi - \frac{2}{3}\cos^3 \pi\right) - \left(0 \cdot 1 - \frac{2}{3}\right)$$

$$= 3\pi + \frac{2}{3}$$