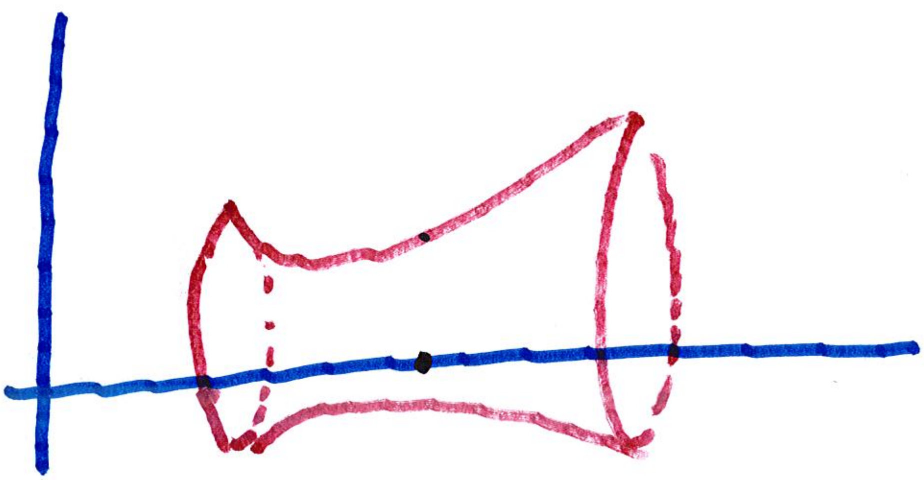
Surfaces of Revolution.

We can rotate a curve

$y = f(x)$ about the x-axis



We use the variables

x and θ as parameters.

We let θ be the angle of rotation for $0 \le \theta \le 2\pi$.

and $a \le x \le b$.

Thus,

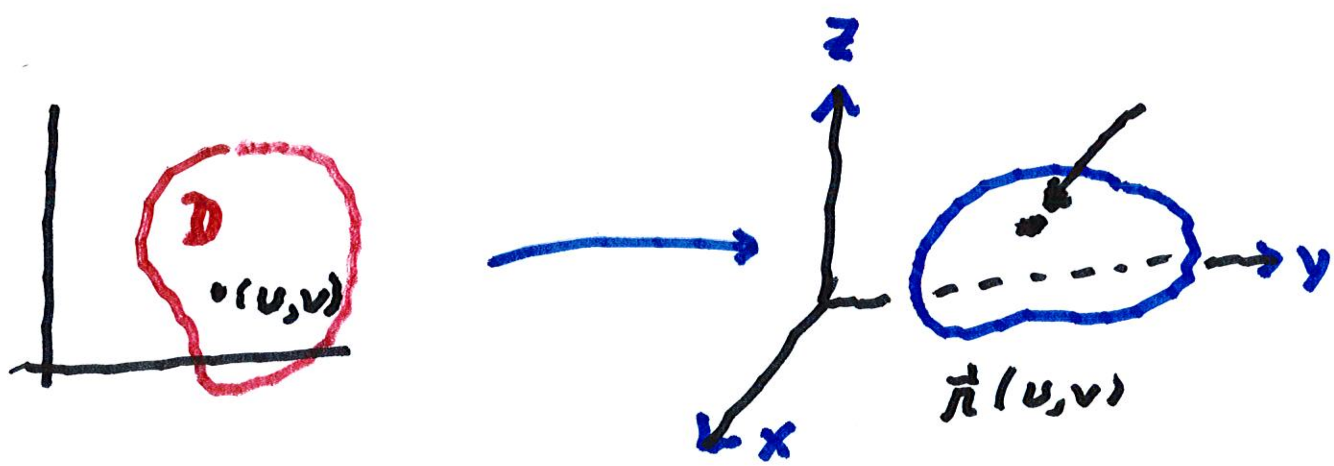
$x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$

or, if $f(x) = \sin x$, $0 \le \theta \le 2\pi$

Recall a surface S can be described by functions

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}.$$

The points on the surface S are defined by various ~~pcc pairs~~ pairs (u,v) in a parameter domain



The points on the sphere of radius a are defined by

$$x(\phi, \theta) = a \sin\phi \cos\theta ,$$

$$y(\phi, \theta) = a \sin\phi \sin\theta ,$$

$$z(\phi, \theta) = z(\phi, \theta) = a \cos\phi .$$

where $0 \le \theta \le 2\pi$,

and $0 \le \phi \le \pi$.

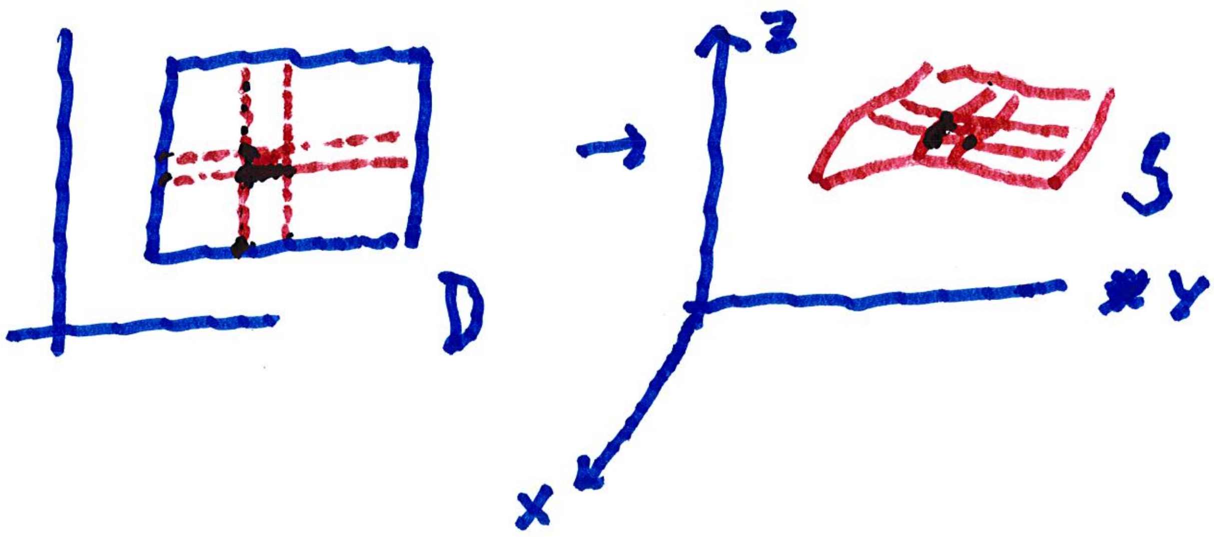
By fixing $v = v_0$, we obtain a function

$$u \to \vec{r}(u, v_0),$$

and by fixing $u = u_0$, we obtain a function

$$v \to \vec{r}(u_0, v).$$

These are called "grid curves"



The small segments are of length Δu and Δv.

They are mapped by $\vec{r}$ to a slight approximation of vectors defined by

~~$\vec{r}'(u_0, v_0)\Delta u$ and $\vec{r}'$~~

$\cancel{\Delta u\, f}\ \vec{r}'$.

$\vec{r}'_u \Delta u$ and $\vec{r}'_v \Delta v$, where

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\vec{j}$$

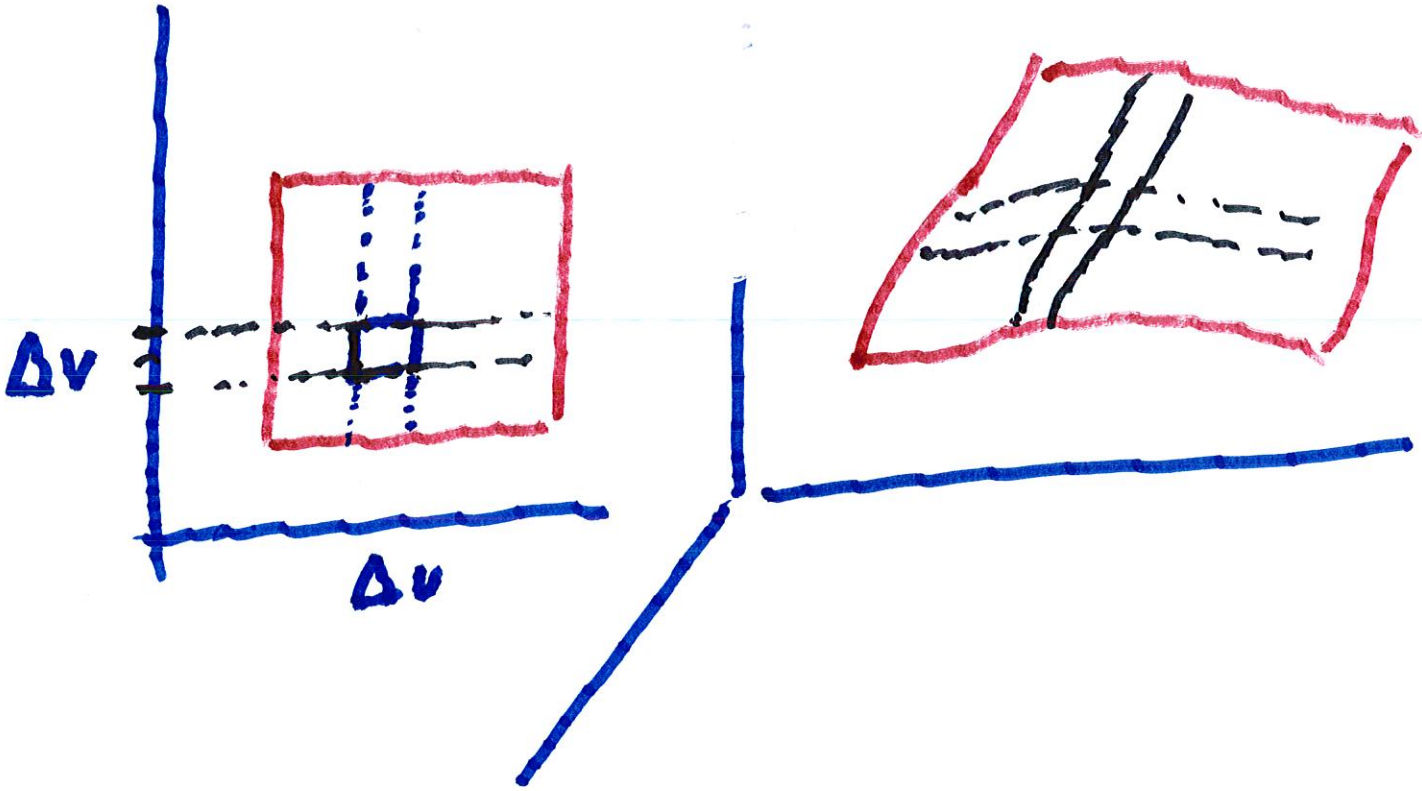
$$+ \frac{\partial z}{\partial u}(u_0, v_0)\,\vec{k}$$

and

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\vec{j}$$

$$+ \frac{\partial z}{\partial v}(u_0, v_0)\,\vec{k} \quad .$$

Clearly $\vec{r}_u$ and $\vec{r}_v$ are both tangent to S



The small rectangles of lengh
Δu and Δv

are mapped to small vectors

$\vec{r}_u \Delta u$ and $\vec{r}_v \Delta v$ which

determine a parallelogram

the ~~angle~~ area of which is

$$|\vec{r}_u \times \vec{r}_v| \Delta u \Delta v.$$

If we subdivide D by

~~One~~ $x_0 < x_1 < \dots < x_m$ $\quad \Delta x = \frac{b-a}{m}$

and

$y_0 < y_1 < \dots < y_n$, $\quad \Delta y = \frac{b-a}{n}$.

then we again obtain that the

approximate surface of S is

$$\sum_{i,j}^{m,n} |\vec{r}_i(P_{ij}) \times \vec{r}_j(P_{ij})| \Delta u \Delta v$$

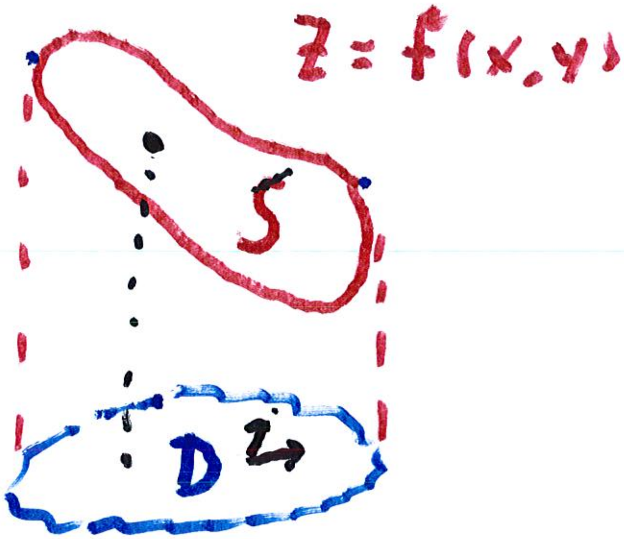
If we let m and $n \to \infty$,

then we see that

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA .$$

Surface Area of the Graph of a Function

Let $S = \{ (x, y, f(x, y)); \ f$ is a C_1 function defined on $D. \}$



The graph of f is defined above. Note that S can be parameterized by

$$(x,y) \to (x, y, f(x,y))$$

Hence, the formula for surface area states that

$$\vec{r}_x = \vec{i} + \left(\frac{\partial f}{\partial x}\right)\vec{k}$$

and

$$\vec{r}_y = \vec{j} + \left(\frac{\partial f}{\partial y}\right)\vec{k}.$$

This implies that

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix}$$

Hence

$$|\vec{n}_x \times \vec{n}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Hence,

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

$$= -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}.$$

Hence

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Therefore

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Ex. Find the surface area of the surface

$$z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right), \quad 0 \le x \le 1$$
$$0 \le y \le 1$$

$$\frac{\partial f}{\partial x} = \sqrt{x} \qquad \frac{\partial f}{\partial y} = \sqrt{y}$$

Hence

$$A(S) = \int_0^1 \int_0^1 \sqrt{1 + x + y}\; dy\, dx$$

$$= \frac{2}{3}\int_0^1 (2+x)^{3/2} - (1+x)^{3/2} \Big|_{y=0}^{y=1} dx$$

$$= \frac{2}{3}\cdot\frac{2}{5}\left[(3)^{5/2} - 2^{5/2}\right]$$

$$- \frac{4}{15}\left[(2)^{5/2} - 1\right]$$

$$= \frac{4}{15}\left[3^{5/2} - 2^{7/2} - 1\right]$$

Ex. Find the surface area of the part of the plane $6 = 3x + 2y + z$ that lies in the first octant

$z = f(x,y)$

$z = 6 - 3x - 2y$

Area of Base $= \frac{2 \cdot 3}{2} = 3$

$f_x = -3, \quad f_y = -2$

$\cancel{A}$ $\frac{\partial z}{\partial x} =$ $\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$

$= \iint_D \sqrt{1 + 9 + 4} = \frac{\sqrt{14}}{3}$