13.1 Vector Functions

A vector function

assigns a vector

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

to each t in the domain of $\vec{r}$. Usually the domain is the set of t such that $f(t)$, $g(t)$, and $h(t)$ are all defined.

Ex. Define

$$\vec{r}(t) = \left\langle \sqrt{t+1},\ \ln(2-t),\ \frac{1}{t^2} \right\rangle$$

Find the domain of $\vec{r}$.

t must satisfy

$t+1 \geq 0$, $2-t > 0$, and $t \neq 0$

or $t \geq -1$, $t < 2$, and $t \neq 0$

$\therefore \text{Dom}(\vec{r}) = \left\{ -1 \leq t < 2,\ t \neq 0 \right\}$

We define

$$\lim_{t \to a} \langle f(t),\ g(t),\ h(t) \rangle$$

$$= \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle,$$

(provided that these

3 limits exist).

Ex. Compute

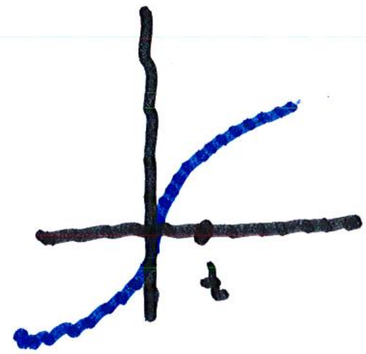
$$\lim_{t\to 0^+} \left\langle t\ln t,\ \sqrt[3]{t}\ ,\ 2t+3 \right\rangle$$

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}$$

$$\underset{\text{L'Hop}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+} (-t)$$

$$= \underline{\underline{0}}$$

$$\lim_{t \to 0^+} \sqrt[3]{t} = 0$$



$$\lim_{t \to 0^+} (2t+3) = 2 \cdot 0 + 3 = 3$$

$$\therefore \lim_{t \to 0^+} \text{limit} = (0, 0, 3)$$

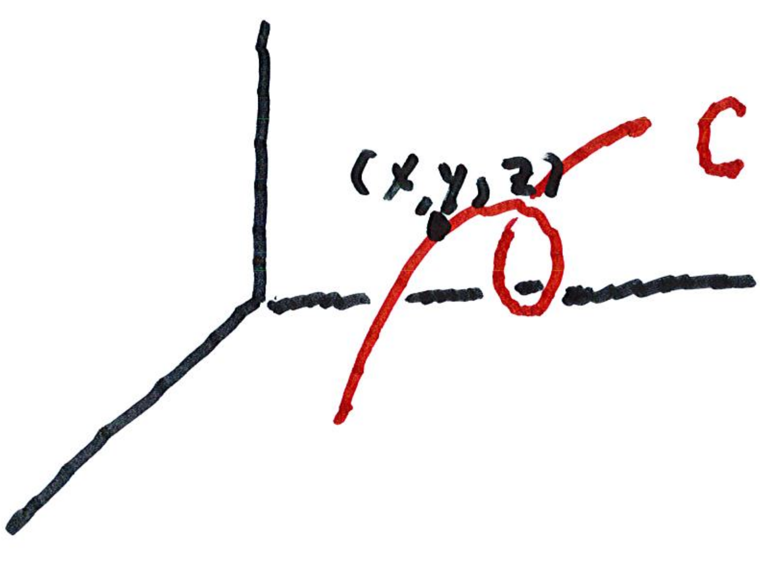
A function $\vec{r}(t)$ is continuous at a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

Suppose that $f(t)$, $g(t)$ and $h(t)$ are all continuous ~~at~~ at all t in an interval I.

We say that a set C of points (x, y, z) is a space curve if for each $(x, y, z) \in C$ there is a number t such that

$x = f(t)$, $y = g(t)$, and $z = h(t)$.



A line segment S is a space curve. Let $r_0 = (x_0, y_0, z_0)$ and $r_1 = (x_1, y_1, z_1)$.

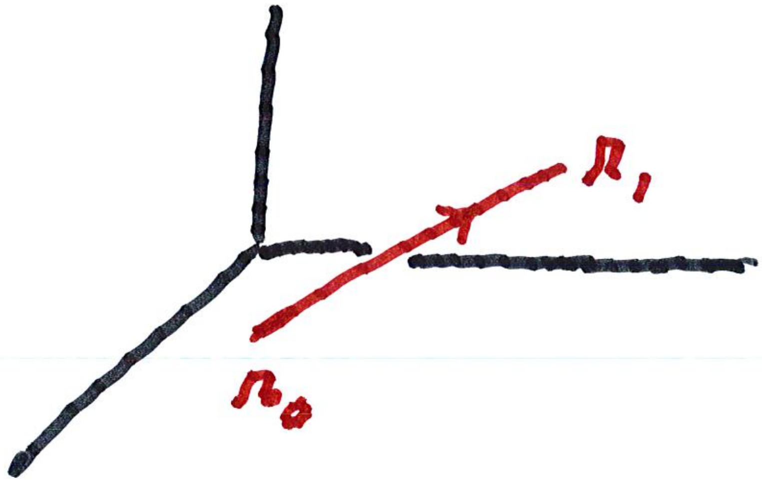
Define $\vec{v} = \overrightarrow{r_0 r_1}$

Let $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, for $0 \le t \le 1$.

$$\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

$\vec{r}(0) = r_0$ and $\vec{r}(1) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)$

$= \vec{r}_1$



$$\therefore \vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

Ex. Parameterize the segment between $(2, 3, 1)$ and $(-1, 9, 10)$

$$\vec{v} = \langle r_1 - r_0 \rangle$$

$$= \langle -1, 9, 10 \rangle - \langle 2, 3, 1 \rangle$$

$$= \langle -3, 6, 9 \rangle$$

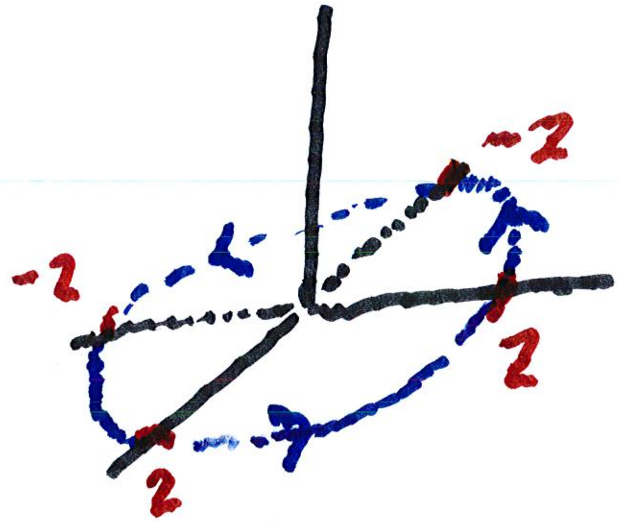
$$\therefore \vec{r}(t) = \langle 2, 3, 1 \rangle + t \langle -3, 6, 9 \rangle$$

Ex. Sketch the curve

$$x = 2\cos t \quad y = 2\sin t, \quad z = t$$

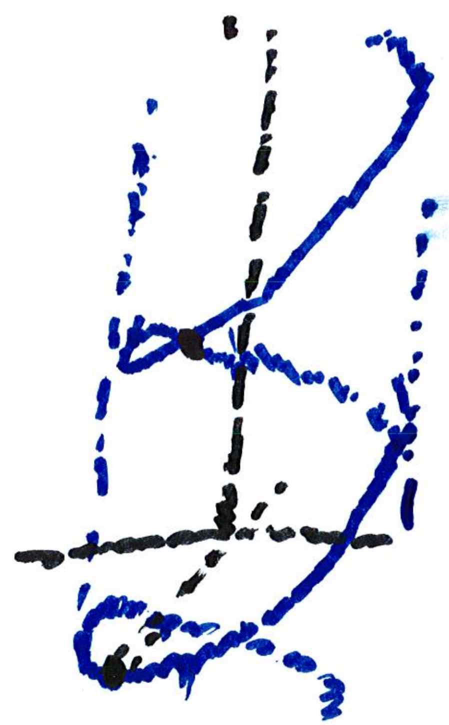
First suppose that

$z(t) = 0$



$(2\cos t, 2\sin t)$ goes in a circle of radius 2.

If $z(t) = t$



We get a helix that winds up around the z-axis.

Ex. Find a parametric curve described by

$$x^2 + 4y^2 = 4 \quad \text{and} \quad z = x^2$$

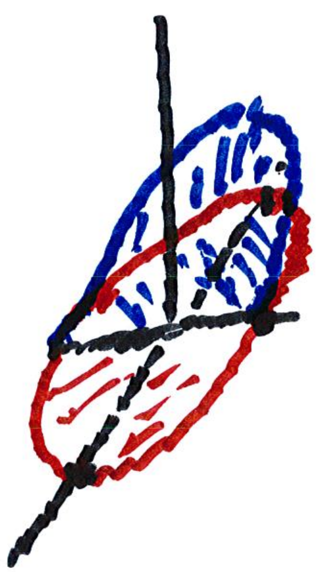
Divide by 4

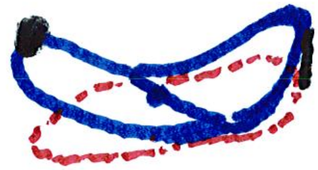
$$\frac{x^2}{4} + y^2 = 1 \quad \text{and} \quad z = x^2$$

Left Curve

Set $x = 2\cos t$ $\rightarrow$ $z = x^2$

and $y = \sin t$ $\quad z = 4\cos^2 t$



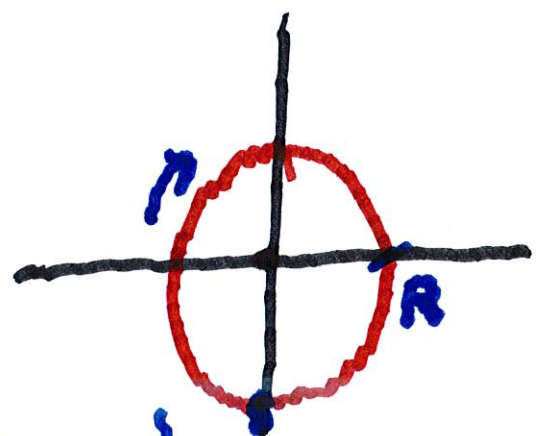


Cycloid (Rolling Wheel)

Imagine a point on a

tire of radius R

at $\theta = \frac{3\pi}{2}$



$$x = R\cos\left(\frac{3\pi}{2} - t\right)$$

$$y = R\sin\left(\frac{3\pi}{2} - t\right)$$

Use formulas for

$\cos(A-B)$ and $\sin(A-B)$:

we get $x = -R\sin t$

and $y = -R\cos t$

If wheel rests on x-axis

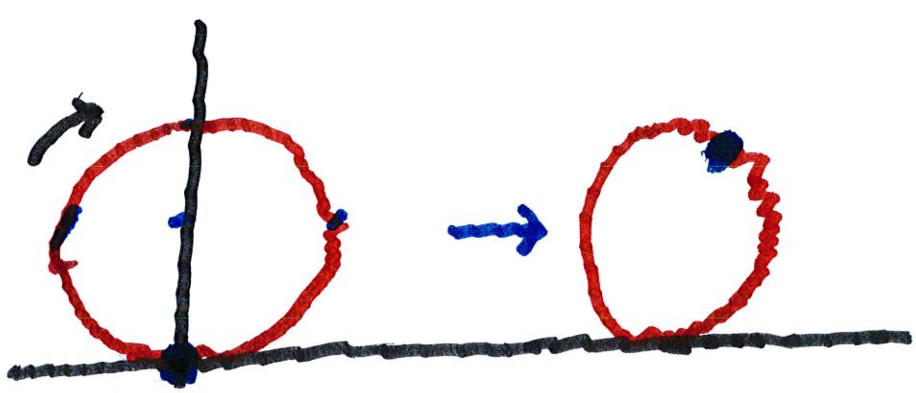
$x = -R\sin t$

$y = R - R\cos t$

After t seconds, the wheel moves Rt to the right;

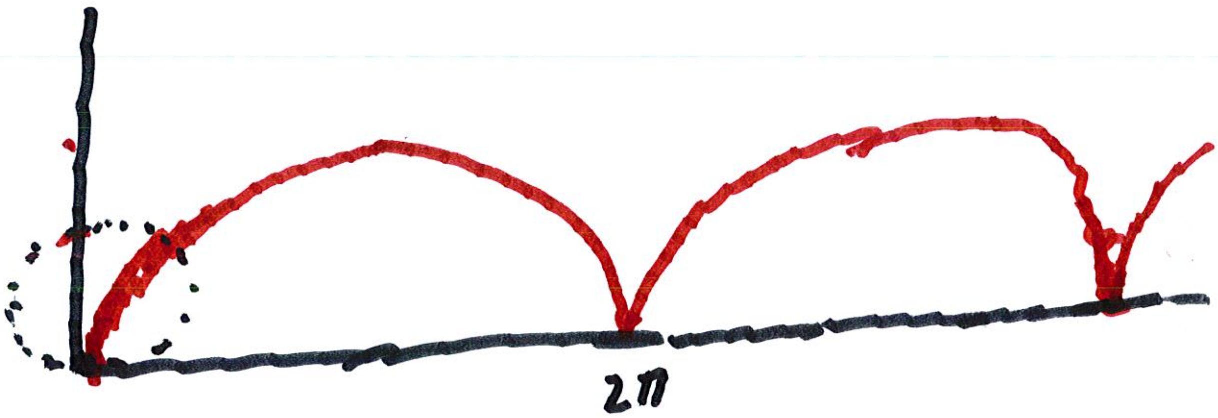
$$x = Rt - R\sin t$$

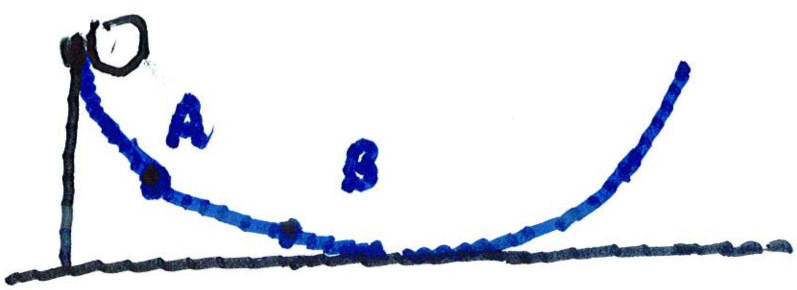
$$y = R - R\cos t$$



When $t = 2\pi$ the wheel moves $2\pi R$ in positive direction

Location of Blue Spot
(Cycloid)



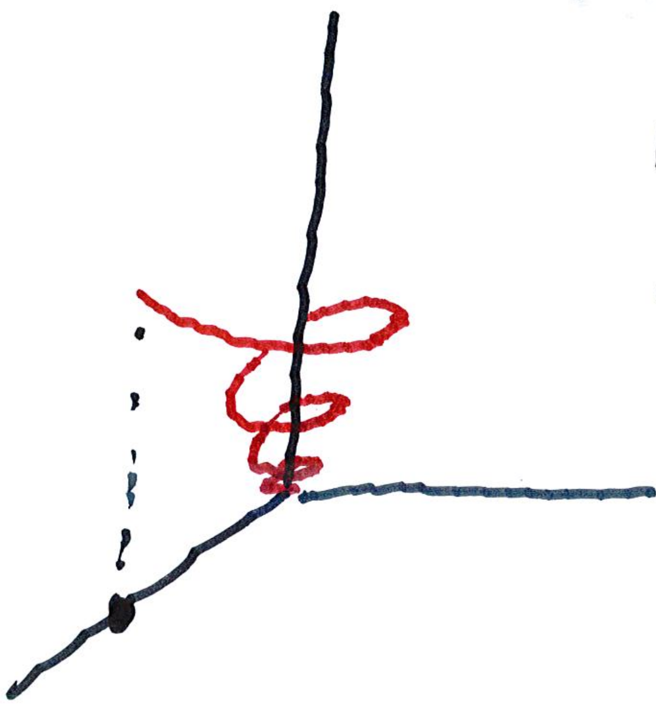


Among all curves joining A and B, the cycloid takes the least time. (Bernoulli)

Ex. Sketch $x = e^{-t}\cos 10t$

$y = e^{-t}\sin 10t$,

when $t > 0$.



Note that the curves makes one spiral when

$10t = 2\pi$, i.e.,

$t = \frac{2\pi}{10}$

And e^{-t} shrinks by a factor of $e^{-2\pi/10}$.

Ex. Two particles have trajectories given by

$\vec{r}_1(t) = \langle t^2, 7t-12t, t^2 \rangle$

$\vec{r}_2(t) = \langle 4t-3, t^2, 5t-6 \rangle$.

Do they collide?

If $\vec{r}_1(t) = \langle x(t), y(t), z(t) \rangle$, note that $x(t) = z(t)$ for all t. At a collision point, it must be that $4t - 3 = 5t - 6$

which implies that $t = 3$.

Note that

$\vec{r}_1(3) = \langle 9, 9, 9 \rangle$

and

$\vec{r}_2(3) = \langle 9, 9, 9 \rangle$

They do collide!

Ex. Find a parametrization of the line sequence from $\vec{a} = \langle 2, 1, -1 \rangle$ to $\vec{b} = \langle 3, 2, -3 \rangle$

Set $\vec{v} = \vec{b} - \vec{a}$

$= \langle 1, 1, -2 \rangle.$

Set $\vec{r}(t) = \langle 2, 1, -1 \rangle + t \langle 1, 1, -2 \rangle,$

for $0 \le t \le 1.$

Ex. Find a vector function that represents the intersection of the curves

$z = 4x^2 + y^2$ and $y = x^2$.

paraboloid

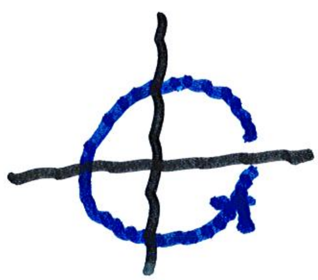
parabolic cylinder

Set $x = t$, $y = t^2$, $z = 4t^2 + t^4$

for all t.

Ex. Find a vector function that represents:

$x^2 + y^2 = 4$, $z = xy$



$x = 2\cos t$, $y = 2\sin t$

$z = 4\cos t \sin t$.