13.2 Derivatives and Integrals of Vector Fcns.

Suppose f(t) and g(t) are differentiable at t. Then

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \frac{\langle f(t+h), g(t+h) \rangle - \langle f(t), g(t) \rangle}{h}$$

$$= \left\langle \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h} \right\rangle$$

which converges to

$\langle f'(t), g'(t) \rangle$, as $h \to 0$

$\therefore \vec{r}'(t) = \langle f'(t), g'(t) \rangle$

Similarly, if

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Ex. If $\vec{r}(t) = \langle e^{2t}, t^2, e^{-t} \rangle$,

find $\vec{r}\,'(t)$ and $\vec{r}\,'(0)$.

$$\vec{r}\,'(t) = \langle 2e^{2t}, 2t, -e^{-t} \rangle$$

$$\vec{r}\,'(0) = \langle 2, 0, -1 \rangle$$

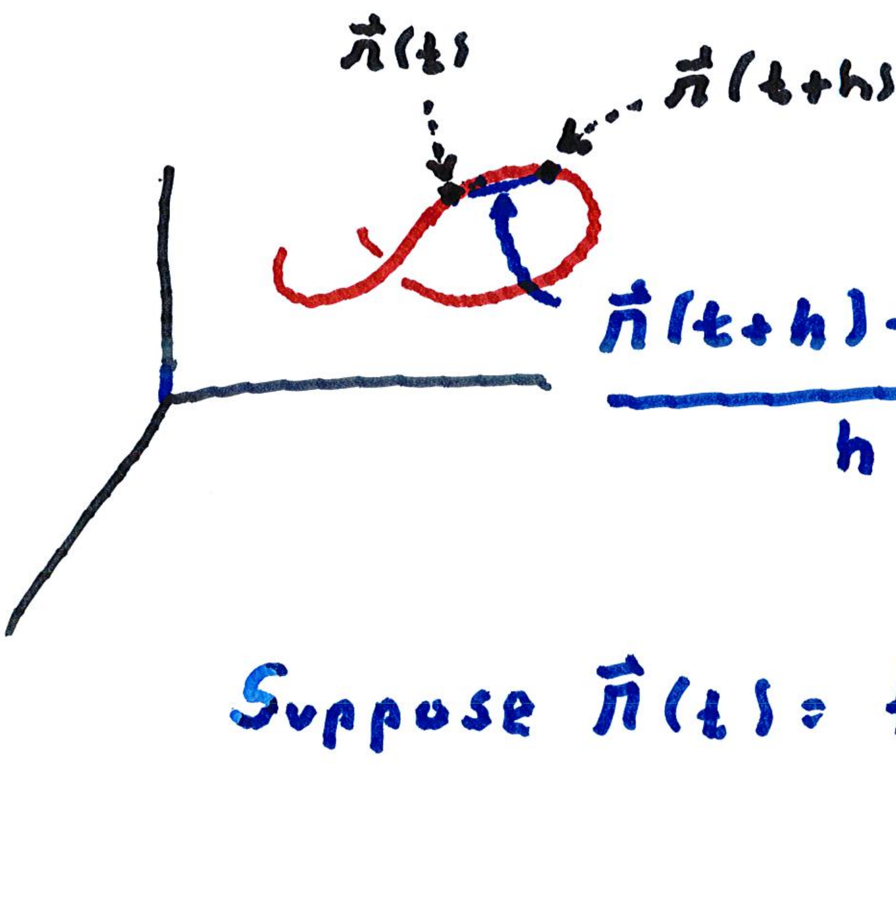
Ex. If $\vec{r}(t) = \langle \tan t, \sec t, \frac{1}{t^2} \rangle$

find $\vec{r}\,'(t)$

$$\vec{r}'(t) = \left\langle \sec^2 t,\ \sec t \tan t,\ \frac{-2}{t^3} \right\rangle$$

$$\left(\text{as long as } \cos t \neq 0 \text{ and } t \neq 0 \right)$$

Geometric Meaning of $\vec{r}'(t)$



$$\frac{\vec{r}(t+h) - r(t)}{h}$$

Suppose $\vec{r}(t) = t\vec{a} + \vec{b}$.

$$\Rightarrow \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \frac{\left((t+h)\vec{a} + \vec{b}\right) - \left(t\vec{a} + \vec{b}\right)}{h}$$

$$= \frac{h\vec{a}}{h} = \vec{a}.$$

Thus $\vec{a}$ gives the direction and magnitude of motion.

In general, if $\vec{r}$ is differentiable at t, then $\vec{r}\,'(t)$ is called the tangent vector to the curve C at the point $P = \vec{r}(t)$

If $\vec{r}\,'(t) \neq 0$, we

define $T(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

to be the <u>unit tangent vector</u>

of the curve C at $\vec{r}(t)$.

T(t) only gives the

direction of the tan line.

Ex. For the curve C parameterized by

$\vec{r}(t) = (1-2t)\vec{i} + \sqrt{t}\,\vec{j}$, $t \geq 0$,

find the unit tangent vector T at $(-1, 1)$.

We need $1-2t = -1$

or $2t = 2 \rightarrow \underline{t=1}$

$$\vec{r}'(t) = -2\vec{i} + \frac{1}{2\sqrt{t}}\vec{j}$$

$$= -2\vec{i} + \frac{1}{2}\vec{j} \qquad (t=1)$$

$$T(1) = \frac{-2\vec{i} + \frac{1}{2}\vec{j}}{\sqrt{4 + \frac{1}{4}}}$$

$$= \frac{-2\vec{i} + \frac{1}{2}\vec{j}}{\frac{\sqrt{17}}{2}} = \frac{-4}{\sqrt{17}}\vec{i} + \frac{\vec{j}}{\sqrt{17}}$$

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

then $\vec{r}\,''(t) = \langle f''(t), g''(t), h''(t) \rangle$

Differentiation Rules

Suppose that $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions and $f(t)$ is a real-valued function. Then

1. $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t)$

2. $\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c\,\vec{u}'(t)$

3. $\frac{d}{dt}\left(f(t)\,\vec{u}(t)\right) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

4 $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

5. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t)$

$+ \vec{u}(t)\times\vec{v}'(t)$

6. $\frac{d}{dt}\left[\vec{v}(f(t))\right] = f'(t)\,\vec{v}'(f(t))$

(Chain Rule)

Ex. Suppose that $|\vec{r}(t)| = c$

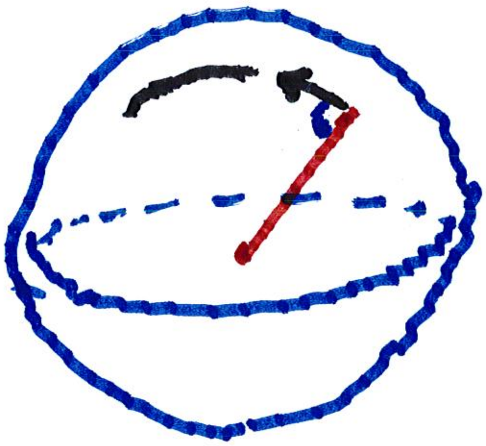
$\Rightarrow |\vec{r}(t)|^2 = c^2$

$\vec{r}(t) \cdot \vec{r}(t) = c^2$ Now diff:

$\therefore\ 0 = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$

$= 2\,\vec{r}(t) \cdot \vec{r}\,'(t)$

Then $\vec{r}\,'(t)$ is $\perp$ to $\vec{r}(t)$



Integrals. If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

We can also compute indefinite integrals

If $\vec{r}(t) = 3t^2\vec{i} + \cos 2t\,\vec{j} + \frac{1}{t^2+1}\vec{k}$

$$\int \vec{r}(t)\,dt = t^3\vec{i} + \frac{\sin 2t}{2}\vec{j} + \tan^{-1}(t)\,\vec{k} + \vec{C},$$

where $\vec{C} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Ex. Find $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\vec{j} + \frac{2t}{1+t^2}\vec{k} \right) dt$

$$\int_0^1 \frac{4}{1+t^2}\,dt = 4\tan^{-1}t \Big|_0^1 = 4\cdot\frac{\pi}{4} - 0 = \pi$$

$$\int_0^1 \frac{2t}{1+t^2}\,dt = \ln(1+t^2)\Big|_0^1 = \ln 2$$

$$\therefore \int_0^1 \left(\frac{4}{1+t^2}\,\vec{j} + \frac{2t}{1+t^2}\,\vec{k} \right) dt$$

$$= \pi \vec{j} + \ln 2\, \vec{k}$$

Ex. Find the point on the curve

$$\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ e^t \rangle,$$

where the tangent

line is parallel to the plane

$\sqrt{3}\,x + y = 1$.

The normal to the plane is

$\langle \sqrt{3}, 1, 0 \rangle$. The tangent

vector to the curve at time t

is $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$

We need $\vec{r}\,'(t) \cdot \langle \sqrt{3}, 1, 0 \rangle = 0$

or

$$-2\sqrt{3}\,\sin t + 2\cos t = 0$$

$$\rightarrow \quad -2\sqrt{3}\,\tan t + 2 = 0$$

$$\rightarrow \quad \tan t = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \vec{r}\left(\frac{\pi}{6}\right) = \left\langle \sqrt{3},\ 1,\ e^{\pi/6} \right\rangle$$

Ex. Find a vector eq'n for the tangent line to the curve of intersection of the cylinders

Ex. Find the tangent line

of the curve

$x = 1 + 2\sqrt{t}$, $y = t^3 - t$, $z = t^3 + t$

at the point (3, 0, 2).

Express the line in

parametric equations.

$x' = 2 \cdot \frac{1}{2} \frac{1}{\sqrt{t}}$, $y' = 3t^2 - 1$, $z' = 3t^2 + 1$

If $1 + 2\sqrt{t} = 3$, then $t = 1$.

Note that $y(1) = 1^3 - 1 = 0$

and $z(1) = 1^3 + 1 = 2$.

Note also that

$x'(1) = 1$, $y' = 2$, and $z'(1) = 4$.

$\therefore$ tangent line is

$r'(1) = (3, 0, 2) + t(1, 2, 4)$

In parametric equations :

$x = 3 + t$, $y = 2t$, $z = 2 + 4t$

Compute

$$\int \left(\sec^2 t \, \vec{i} + t(t^2+1)^3 \, \vec{j} + t^2 \ln t \, \vec{k} \right)$$

1. $\int \sec^2 t \, dt = \tan t \, \vec{i}$

2. $\int t(t^2+1)^3 \, dt = \int \frac{(t^2+1)^3}{2} 2t \, dt$

$= \frac{(t^2+1)^4}{2 \cdot 4}$

$u = t^2 + 1$

$du = 2t \, dt$

3 $\int t^2 \ln t \, dt =$

$u = \ln t \quad dv = t^2 \, dt$

$du = \frac{dt}{t} \quad v = \frac{t^3}{3}$

$$= \ln t \cdot \frac{t^3}{3} - \int \frac{t^3}{3} \frac{dt}{t}$$

$$= \ln t \cdot \frac{t^3}{3} - \frac{1}{9} t^3$$

∴ Original Vector integral is:

$$\tan t \, \vec{i} + \frac{(t^2+1)^4}{8} \vec{j} + \left(\frac{\ln t \cdot t^3}{3} - \frac{t^3}{9} \right) \vec{k}$$

Recall that if $\vec{r}(t)$ is a smooth vector function, then

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

is the derivative. At each t, it gives the direction and the magnitude of the motion. Find $\vec{T}(t)$.

Ex. If $\vec{r}(t) = \langle 2t+1, t^2, 3t \rangle$

$$\vec{r}\,'(t) = \langle 2, 2t, 3 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4 + 4t^2 + 9}$$

$$\vec{T}(t) = \left\langle \frac{2}{\sqrt{13+4t^2}}, \frac{2t}{\sqrt{13+4t^2}}, \frac{3}{\sqrt{13+4t^2}} \right\rangle$$

As we shall see,

$|\vec{r}\,'(t)|$ = rate of change of motion at time t.

If $\vec{r}(t) = \left\langle t^2 - 1,\ 2t + 1,\ t^2 + t \right\rangle$

Find $T(t)$

$\vec{r}\,'(t) = \langle 2t,\ 2,\ 2t + 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 4 + 4t^2 + 4t + 1}$

$= \sqrt{8t^2 + 4t + 5}$

$$T(t) = \frac{\langle 2t,\ 2,\ 2t+1 \rangle}{\sqrt{8t^2 + 4t + 5}}$$

Sometimes we need the

unit tangent vector

$$T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

As a vector it gives the

direction, but not the

magnitude, since $|T(t)| = 1$

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

then $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

and $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

$\vec{r}''(t)$ is the second derivative

Ex. Find an expression for

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\left(\vec{v}(t)\times\vec{w}(t)\right)\right]$$

By Leibnitz' Rule

$$= \vec{u}\,'(t)\cdot\left(\vec{v}(t)\times\vec{w}(t)\right)$$

$$+ \vec{u}(t)\cdot\left(\vec{v}\,'(t)\times\vec{w}(t)\right)$$

$$+ \vec{u}(t)\cdot\left(\vec{v}(t)+\vec{w}\,'(t)\right)$$