13.4 Velocity and Acceleration

Suppose the position of a particle at time t is $\vec{r}(t)$.

After h seconds, the distance it travels is $|\vec{r}(t+h) - \vec{r}(t)|$

$\therefore$ The speed is roughly

$$\frac{|\vec{r}(t+h) - \vec{r}(t)|}{h} = \left| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right|$$

As $h \to 0$, the limit is the speed

$|\vec{r}\,'(t)|$. If we focus on

$\vec{r}\,'(t)$, we get a vector

that gives the direction of

motion and its speed.

Thus, $\vec{r}\,'(t)$ is the velocity of

the particle.

Similarly, $\vec{v}'(t) = \vec{a}(t)$ is

the acceleration of the particle.

We can rewrite this as

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

Also $|\vec{v}(t)| = |\vec{r}'(t)|$ is

the speed.

Ex. Suppose $\vec{r}(t) = \langle 2t+1, 3t^2 \rangle$

is the position at time t.

Compute it's velocity, speed, and acceleration at time t:

$\vec{r}\,'(t) = \langle 2, 6t \rangle$ = velocity $\vec{v}(t)$

$|\vec{v}(t)| = \sqrt{4 + 36t^2}$ = speed

$\vec{a}(t) = \vec{r}\,''(t) = \langle 0, 6 \rangle$ = acceleration

Ex. If $\vec{r}(t) = (t^2+1)\vec{i} - t^3\vec{j} + 2t^2\vec{k}$

is the position of a particle,

compute the velocity vector, the speed, and the acceleration

$$\vec{v}(t) = \vec{r}\,'(t) = 2t\,\vec{i} - 3t^2\,\vec{j} + 4t\,\vec{k}$$

$$\therefore \text{Speed} = \sqrt{4t^2 + 9t^4 + 16t^2}$$

$$= \sqrt{20t^2 + 9t^4}$$

$$\vec{a}(t) = \vec{r}\,''(t) = 2\,\vec{i} - 6t\,\vec{j} + 4\,\vec{k}$$

is the acceleration

We can go in the other direction and assume we know the acceleration:

Ex. Suppose the acceleration of a particle is

$$\vec{a}(t) = (2t-1)\vec{i} + 3t^2\vec{j} + (2-t)\vec{k}$$

and also that $\vec{v}(0) = \vec{i} + 2\vec{j}$

and $\vec{r}(0) = \vec{j} - \vec{k}$

These are the initial conditions.

Find the position $\vec{r}(t)$.

$$\vec{v}'(t) = \vec{a}(t), \quad \text{so } \vec{v}(t) = \int \vec{a}(t)\,dt$$

Hence $\vec{v}(t) = (t^2 - t)\vec{i} + t^3\vec{j} + \left(2t - \frac{t^2}{2}\right)\vec{k}$

$+ \vec{C}$

$\vec{v}(0) = \vec{0} + \vec{C}$

$\therefore \vec{C} = \vec{i} + 2\vec{j}$

$\rightarrow \vec{v}(t) = (t^2 - t + 1)\vec{i} + (t^3 + 2)\vec{j}$

$+ \left(2t - \frac{t^2}{2}\right)\vec{k}$

Similarly,

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\frac{t^4}{4} + 2t\right)$$

$$+ \left(t^2 - \frac{t^3}{6}\right)\vec{k} + \vec{C}$$

$\vec{C} = \vec{r}(0) = \vec{j} - \vec{k}$, so

$$\vec{r}(t) = \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\frac{t^4}{4} + 2t + 1\right)\vec{j}$$

$$+ \left(t^2 - \frac{t^3}{6} - 1\right)\vec{k}$$

In general, many problems in physics start with the force $\vec{F}$ known. Since $\vec{F} = m\vec{a}$, this leads to knowing $\vec{a}$.

Ex Suppose $\vec{r}(t) = a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}$

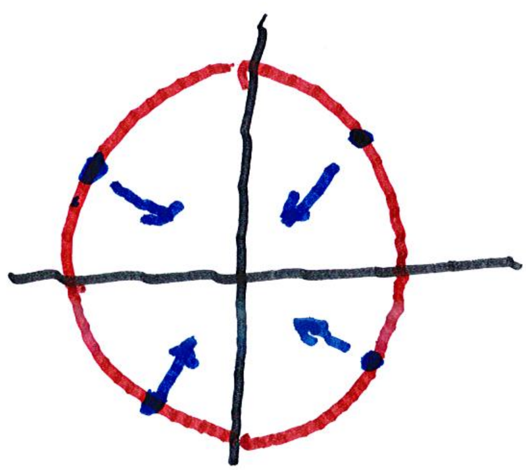
Find $\vec{a}(t)$.

$$\vec{r}\,'(t) = -a\omega\sin\omega t\,\vec{i} + a\omega\cos\omega t\,\vec{j}$$

Hence .

$$\vec{a}(t) = -a\omega^2 \cos\omega t\,\vec{i} - a\omega^2 \sin\omega t\,\vec{j}$$

$$= -\omega^2 \left(a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}\right)$$



We see that

$$\vec{a}(t) = -\omega^2 \vec{r}(t)$$

This indicates that

acting on the particle points inward.

Motion on earth.

Suppose a projectile is fired

at $t=0$, with $\vec{r}(0) = \vec{0}$

It's initial velocity is

(1) $\vec{v}(0) = V_0 \left(\cos\alpha \,\vec{i} + \sin\alpha \,\vec{j} \right)$

The force acting on the particle

is $\vec{F} = m\vec{a} = -mg\vec{j}$

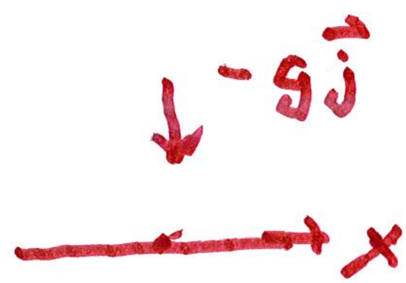
$\therefore$ acceleration $= \vec{a} = -g\vec{j}$

$$\rightarrow \vec{v}(t) = -gt\vec{j} + \vec{C} = \underline{-gt\vec{j} + \vec{v}(0)}$$

$$\rightarrow \vec{r}(t) = \underline{\frac{-gt^2}{2}\vec{j} + t\vec{v}(0) + \vec{D}}.$$

Since $\vec{r}(0) = \vec{0}$, $\vec{D} = 0$.

Recall (1), we get



$$\vec{r}(t) = v_0 t\cos\alpha\,\vec{i} + \left(v_0 t\sin\alpha - \frac{gt^2}{2}\right)\vec{j}$$

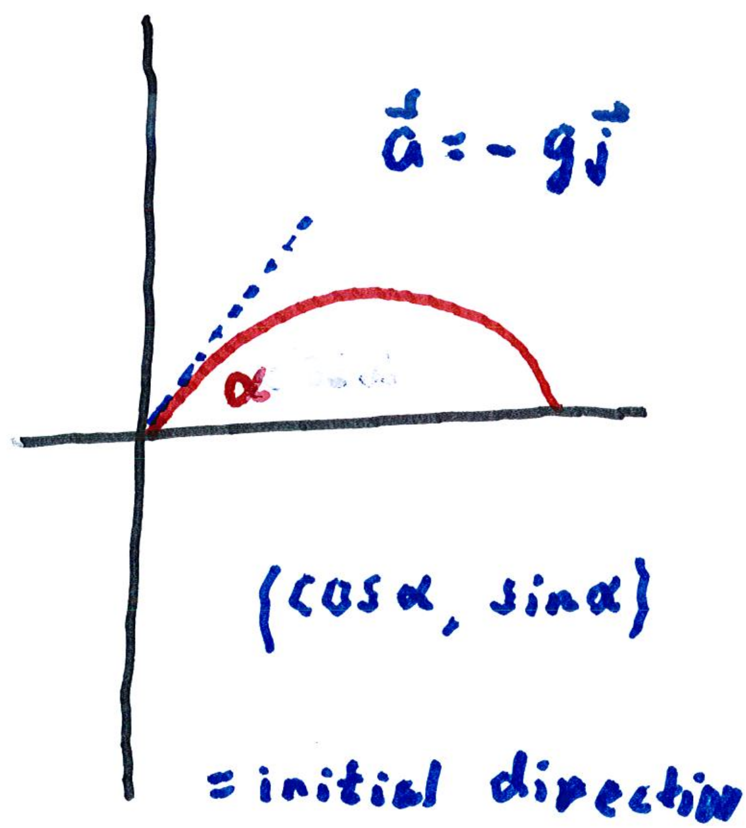
Since $\vec{v}(0) = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$

We obtain

$$\boxed{\begin{aligned} x(t) &= V_0 t \cos\alpha \, \vec{i} \\ y(t) &= \left(V_0 t \sin\alpha - \frac{gt^2}{2} \right) \vec{j} \end{aligned}}$$

Ex. (Re

Motion in the plane.



Tangential and Normal Components of Acceleration

Using the formula in 13.3.9, one can show that the acceleration $\vec{a}$ satisfies

$$\vec{a} = v'T + \kappa v^2 N,$$

where κ = curvature, and T and N are the tangential and normal components.

Thus, one can accelerate by increasing speed in the $\vec{T}$ direction and by curving in the direction $\vec{N}$.

Review. Find the length L of the path

$$\vec{r}(t) = 12t\vec{i} + 12t^{3/2}\vec{j} + 3t^2\vec{k}$$

$$r'(t) = 12\vec{i} + 12t\vec{j} + 6t\vec{k}$$

$$|r'(t)| = 6\sqrt{4 + 4}$$

$$\therefore \; |r'(t)| = \sqrt{12^2 + 12^2 t + 6^2 t^2}$$

$$= 6\sqrt{4 + 4t + t^2}$$

$$\therefore L = 6\int_0^1 \sqrt{(t+2)^2}\,dt$$

$$= 6\int_0^1 (t+2)\,dt$$

$$= 6\left(\frac{t^2}{2} + 2t\right)\Big|_0^1 = 6 \cdot \frac{5}{2} = 15$$