14.1 Functions of Several Variables

We often think of a surface

as $z = x^2 y$ or $z = y + x^2$

and we write

$$f(x,y) = x^2 y \quad \text{or} \quad g(x,y) = y + x^2.$$

Instead of 1 variable x,

$f(x,y)$ depends on 2 variables

Def'n A function f of two variables is a rule that assigns to each ordered pair of numbers (x, y) in a set D a unique real number denoted by $f(x,y)$. D is the domain of f and its range is the set of values that f takes, i.e.,

$$R_f = \{ f(x,y) \mid (x,y) \in D \}$$

If the domain D is not specified, then the domain is the set of (x,y) such that $f(x,y)$ is well-defined.

Ex. Find the domain and range of $f(x,y) = \dfrac{\ln(x^2+y-2)}{x-2}$

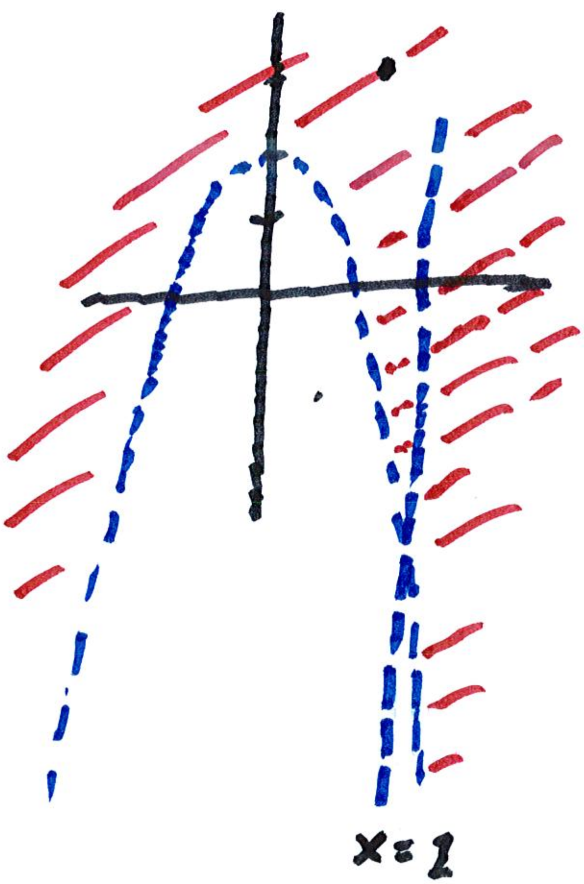
For $\ln(x^2+y-2)$, we need

$x^2 + y - 2 > 0$, i.e.,

$y > 2 - x^2$.

For the denominator $x-2$,

we need $x-2 \neq 0$ or $x \neq 2$.



$D: \{ (x,y) \mid y > 2 - x^2$

and $x \neq 2 \}$

Look at $x=0$

and $y > 2$

$f(0, y) = \dfrac{\ln(y-2)}{-2}$ takes on all values.

$\therefore\ R_f = \{$all real numbers$\}$

Ex. Let $g(x, y, z)$

$= x^3 y^2 z \sqrt{10 - x - y - z}$

Find the domain and range.

We need $10 - x - y - z \geq 0$,

i.e., $\underbrace{x + y + z \leq 10}_{\text{Domain } D}$

For the range, look

at $x = t$, $y = -t$ and $z = 1$

Ex. $P(x,y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$.

We need $x \geq 0$ and $y \geq 0$

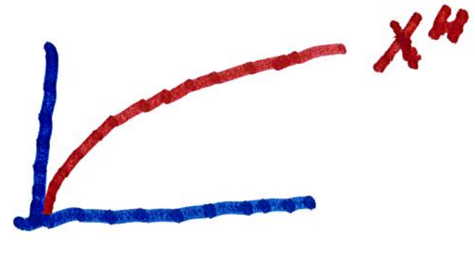
$\therefore D = \{(x,y);\ x \geq 0 \text{ and } y \geq 0\}$

Note that

$P(x,1) = x^{1/4} \geq 0$ for all $x \geq 0$

$\therefore R_p = \{x;\ x \geq 0\}$

Set $y=1 \rightarrow P(x,1) = x^{1/4}$



Ex. If f is a function of

two variables with domain D,

then the graph of f is the

set of all (x, y, z) such that

$z = f(x, y)$ and $(x, y) \in D$.

Ex. Find the

graph of $\longrightarrow$ $z = 2\sqrt{x^2 + y^2}$

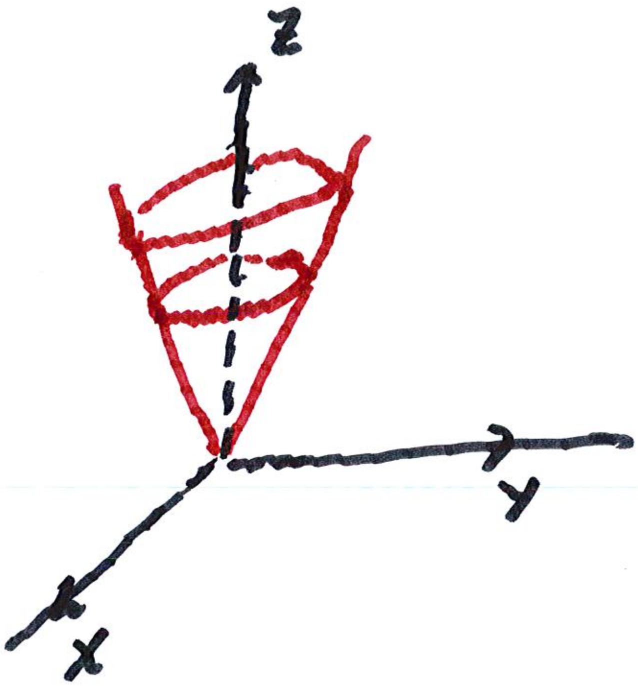
$f(x, y) = 2\sqrt{x^2 + y^2}$

or

$z^2 = 4(x^2 + y^2)$

$z \geq 0$

any $(x, y) \in D_{om}$.



Note that

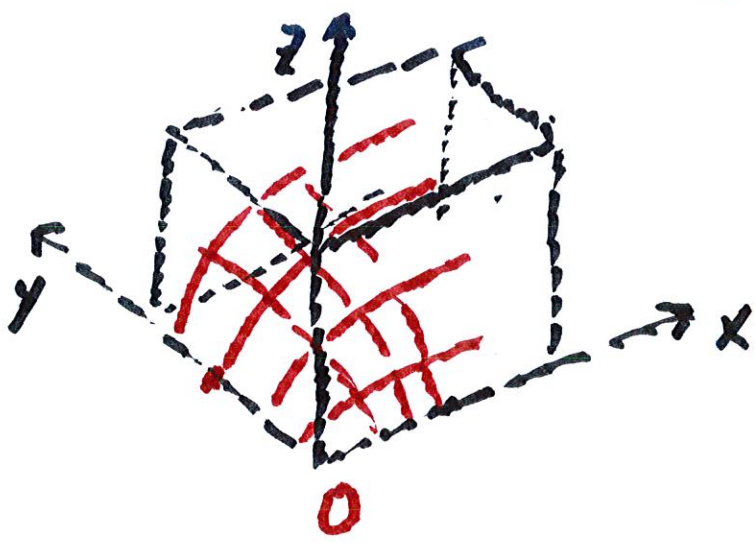
z must be

$z \geq 0$

Sketch the graph of

$f(x, y) = x^{1/4} y^{3/4}$

$z = x^{1/4} y^{3/4}$



Ex. Sketch the graph of

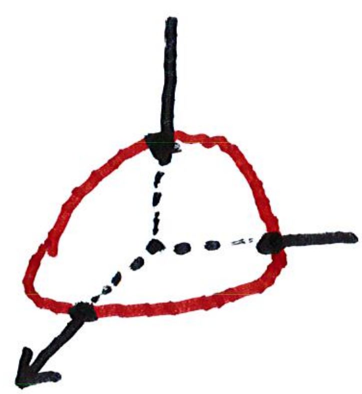
$\sqrt{4-x^2-y^2} \rightarrow z = \sqrt{4-x^2-y^2}$

$\rightarrow z^2 = 4-x^2-y^2$

$\rightarrow x^2+y^2+z^2 = 4$

This is a sphere of radius
hemi

2. Recall $z \geq 0$



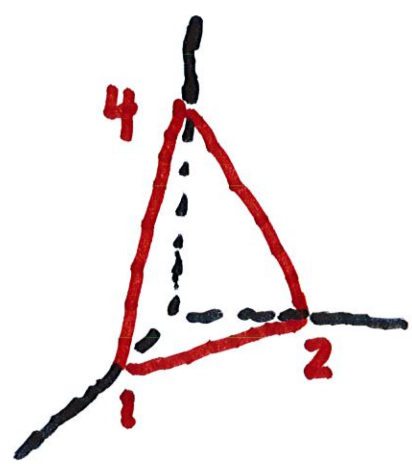
Ex. Find the graph of

$f(x,y) = 4 - 4x - 2y$

$z = 4 - 4x - 2y$

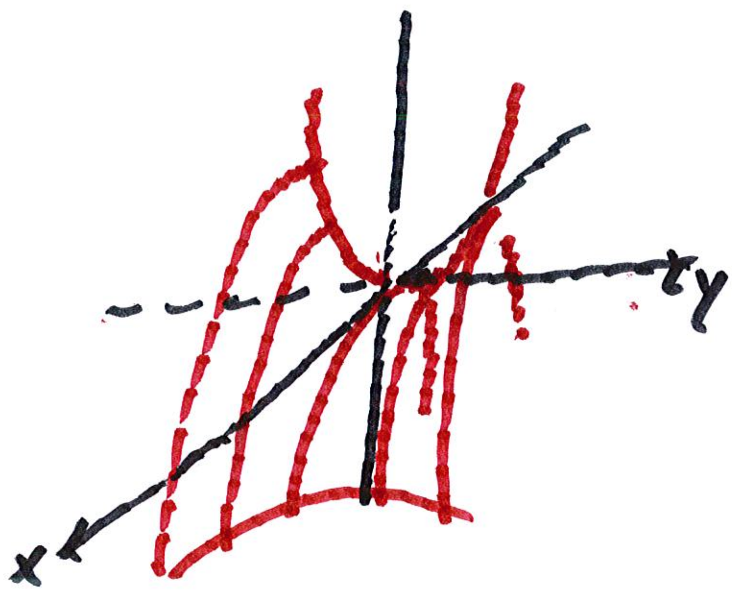
$4x + 2y + z = 4.$

Use x, y, and z intercepts to sketch it



Find Graph of $f(x,y) = y^2 - x^2$

$\rightarrow \quad z = y^2 - x^2$



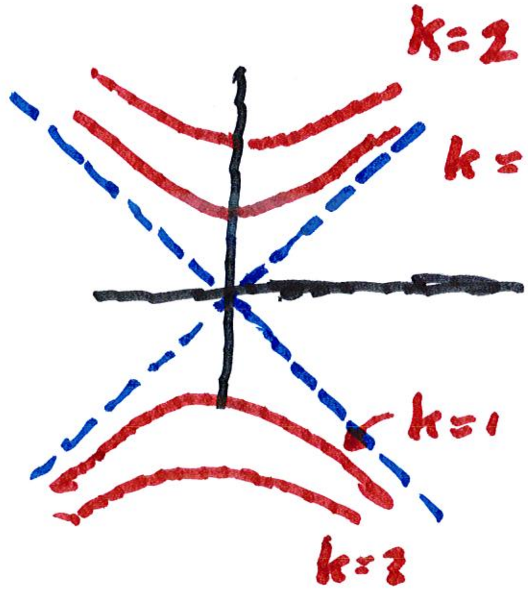
Saddle point at origin.

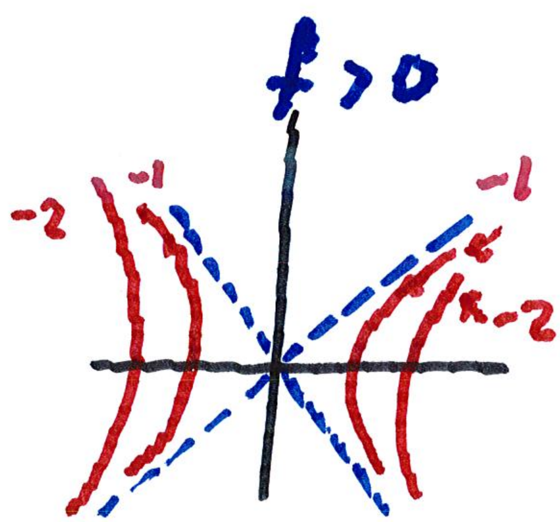
Find the level curves and the graph of $f(x,y) = (1-x-y)^2$

Level Curves of $f(x,y)$

A level curve is = $\{(x,y) \mid f(x,y) = k\}$

Ex. If $f(x,y) = \{x,y \mid y^2 - x^2 = k\}$



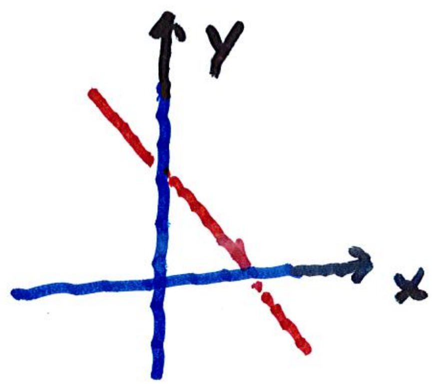


11. Sketch the level curves of $f(x, y) = (x+y-1)^2$

If $k < 0$, $(x+y-1)^2 = k$. NO SOL'N.

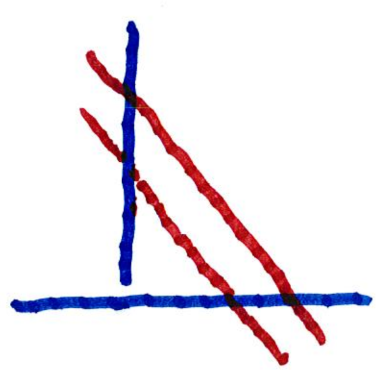
If $k=0$, level curve is

$x+y-1=0$



If $k>0$, we get $(x+y-1)^2 = k$

$x+y-1 = \pm\sqrt{k}$

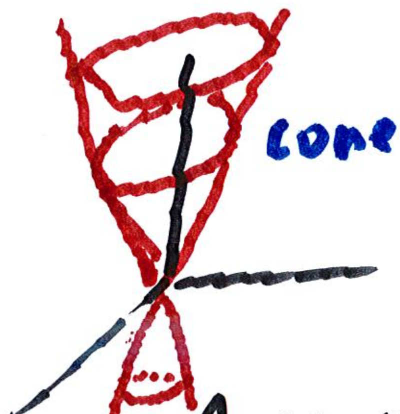


A level surface of a function $f(x, y, z)$ is the set of points (x, y, z) such that $f(x, y, z) = k$.

Sketch the level surface of

$x^2 + y^2 - z^2$

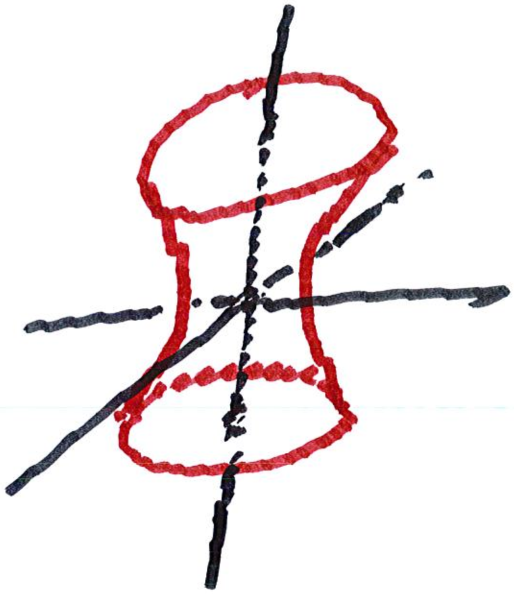
If $k = 0 \to z^2 = x^2 + y^2$



A cone is rotated about the z-axis

If $k > 0$, say $k=1$

$x^2 + y^2 = 1 + z^2$

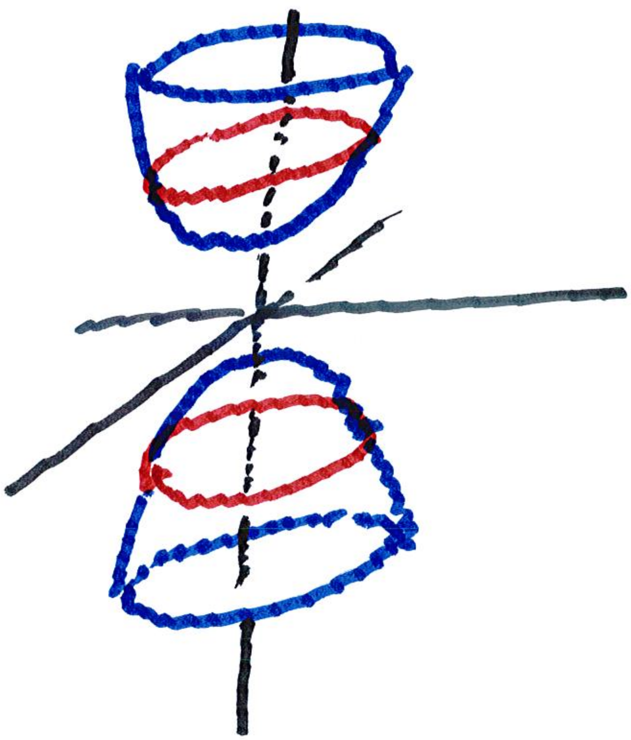


hyperboloid of

1 sheet.

Now suppose that $k < 0$,

say $k = -1$



hyperboloid

of 2 sheets

Ex. Find the level surfaces

(The set of $(x, y, z) \mid x^2 + y^2 + z^2 = k$)

If $k < 0$, no solution at all

If $k = 0$, $x^2 + y^2 + z^2 = 0$

(the origin)

If $k > 0$, $x^2 + y^2 + z^2 = k$

$\rightarrow$ sphere of radius $\sqrt{k}$

1. $z = \sin(xy)$

2. $z = e^x \cos y$

3. $z = \sin(x-y)$

55–58 Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

55. $f(x, y) = xy^2 - x^3$ (monkey saddle)

56. $f(x, y) = xy^3 - yx^3$ (dog saddle)

57. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

58. $f(x, y) = \cos x \cos y$

59–64 Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

59. $z = \sin(xy)$

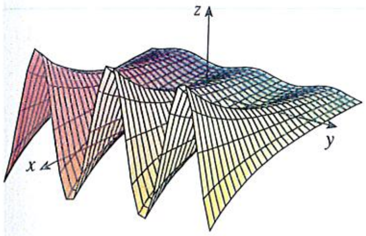
60. $z = e^x \cos y$

61. $z = \sin(x - y)$

62. $z = \sin x - \sin y$

63. $z = (1 - x^2)(1 - y^2)$

64. $z = \dfrac{x - y}{1 + x^2 + y^2}$



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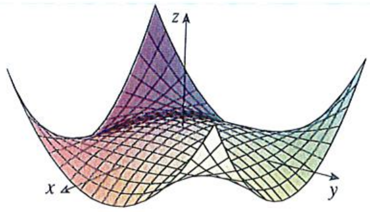

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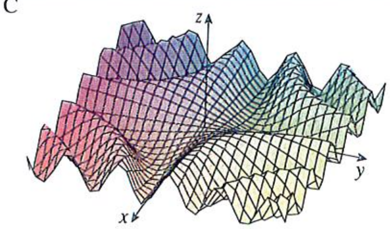
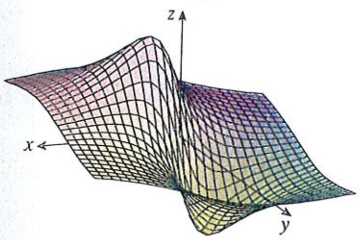
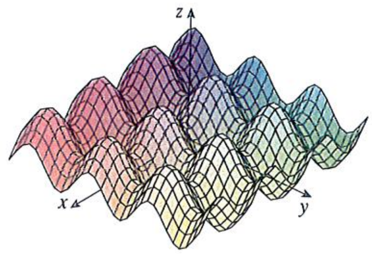
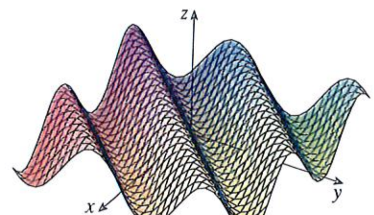


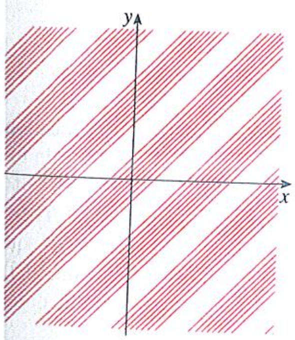


E

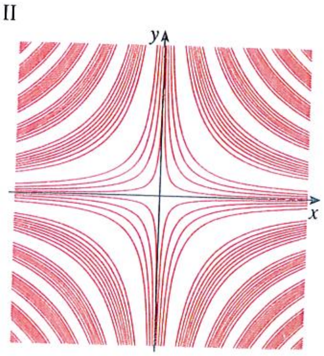


F

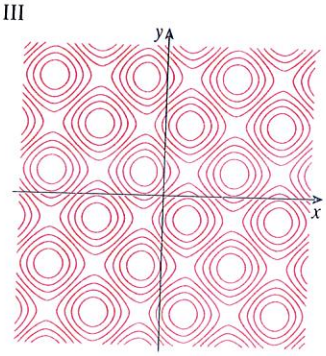


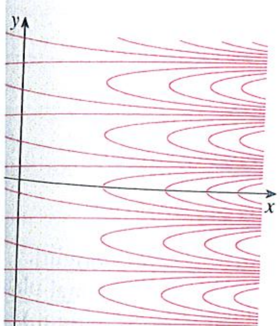


II

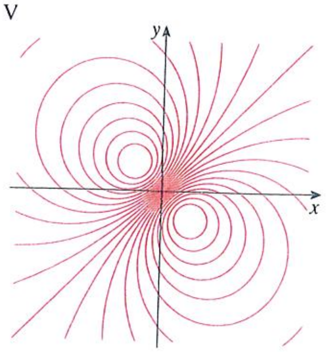


III

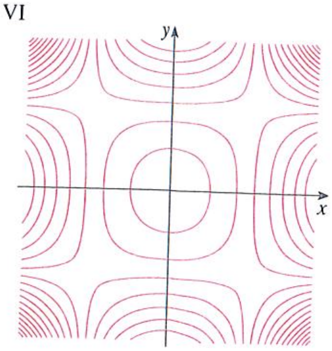




V



VI



The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 4 in Section 12.6. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

$z = 4x^2 + y^2$

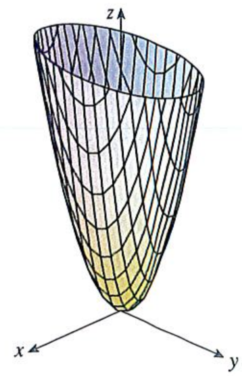
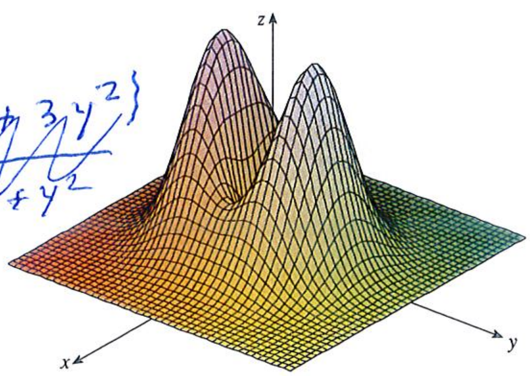


FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$

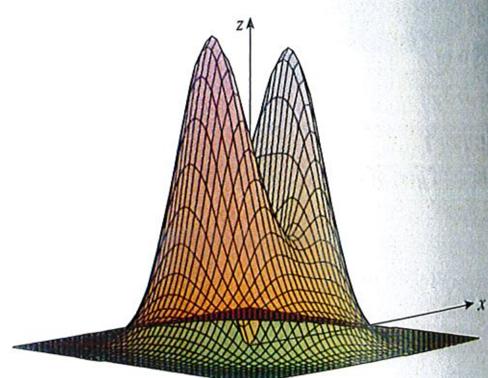
Computer programs are readily available for graphing functions of two variables. In most such programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.

Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from different

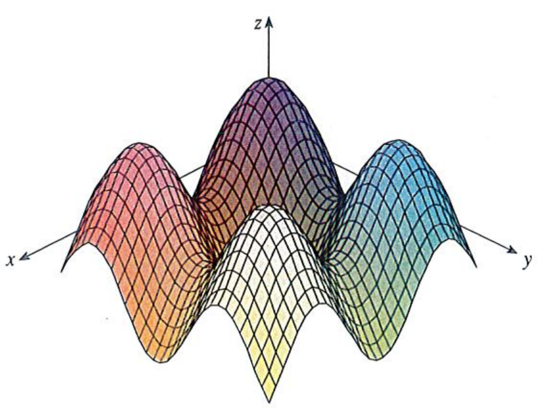
$Z = (x^2 + 3y^2)$ ~~$4x^2 + y^2$~~



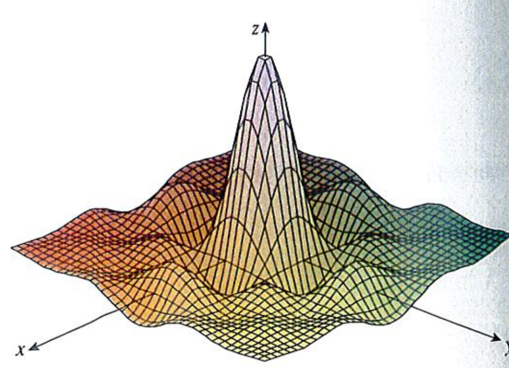
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



(c) $f(x, y) = \sin x + \sin y$



(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

FIGURE 10