Exam 1 Math 341 Name ______________

1. Prove that $n < 2^n$ for all $n \in N$.

We use Induction. The inequality is obvious when $n=1$ $(1 < 2^1)$

Now assume that $n < 2^n$ is true.

Multiply by 2: $2n < 2 \cdot 2^n = 2^{n+1}$.

Note that $n+1 \leq 2n$. We obtain

$n+1 \leq 2n < 2^{n+1}$. Thus

the inequality is true for $n+1$,

By Induction, the inequality is true for all n

2. (a) What is the definition of an upper bound of S?

u is an upper bound of S if

$u \geq s$ for all $s \in S$.

(b) If u is an upper bound of S, under what condition is u a least upper bound?

u must satisfy: If v is also an upper bound of S, then $v \geq u$.

OR: If $\varepsilon > 0$, then there is an element $s_\varepsilon \in S$ such that

$u - \varepsilon < s_\varepsilon$

3. If $S = \{2 - \frac{3}{n} : n \in N\}$, prove that 2 is a least upper bound.

Every element of S is given by

$s = 2 - \frac{3}{n}$ for some $n \in N$.

Note that $2 - \frac{3}{n} < 2$. Thus

$2 \geq s$ for all $s \in S$. $\rightarrow$ 2 is an

upper bound of S

4. Prove that if (x_n) is a convergent sequence, then $\{x_n : n \in N\}$ is bounded.

Pf. We are given that $\lim (x_n) = x$.

If we set $\varepsilon = 1$, then there is a $K \in N$, so if $n \geq K$, then

$|x_n - x| < 1.$

$$\therefore |x_n| = |(x_n - x) + x|$$

$$\leq |x_n - x| + |x| \leq 1 + |x|,$$

for $n \geq K$.

Hence,

$$|x_n| \leq \operatorname{Max}\{|x_1|, \ldots, |x_{K-1}|, 1 + |x|\} = M.$$

5. (a) Define the Nested Interval Property. If $I_n = [a_n, b_n]$ all satisfy $I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$, then there is a number $\eta \in I_n$ for all n.

(b) State the Bolzano-Weierstrass Theorem.

A bounded sequence has a convergent subsequence.

(c) Give the definition of a Cauchy sequence.

(x_n) is Cauchy if for all $\varepsilon > 0$, there is a $K \in \mathbb{N}$, so that if $m \geq K$ and $n \geq K$, then $|x_m - x_n| < \varepsilon$.

6. If $\lim x_n = x$ and $\lim y_n = y$, prove that $\lim(x_n y_n) = xy$.

Note that

$$|x_n y_n - xy| = |x_n(y_n - y) + (x_n - x)y|$$

$$\leq |x_n||y_n - y| + |y||x_n - x| \qquad (1)$$

Recall there is a K_0 so that if $n \geq K_0$, then $|x_n| \leq M_0$.

If $M = \text{Max}\{M_0, |y|\}$, then (1) is bounded by $M|y_n - y| + M|x_n - x|$. (2)

Now choose K_1 so if $n \geq K_1$, then

$$|y_n - y| < \frac{\varepsilon}{2M}, \quad |x_n - x| < \frac{\varepsilon}{2M}$$

Hence (2) is bounded by

$$M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

This implies that $\lim (x_n y_n) = xy$

7. Suppose that (x_n) is a bounded increasing sequence. Prove that there is a number $\tilde{x}$ such that $\lim x_n = \tilde{x}$.

Let $\tilde{x} = \sup\{x_n : n \in N\}$. Choose any $\varepsilon > 0$. Then there is $K \in N$ so that $\tilde{x} - \varepsilon < x_K \leq x_n \leq \tilde{x} < \tilde{x} + \varepsilon$. The second inequality follows since (x_n) is increasing, and the third since $\tilde{x}$ is an upper bound. Hence,

$\tilde{x} - \varepsilon < x_n < \tilde{x} + \varepsilon$. By subtracting, $-\varepsilon < x_n - \tilde{x} < \varepsilon$. This implies

$\lim x_n = \tilde{x}$.

8. (a)State the Squeeze Theorem. Suppose that

$x_n \le y_n \le z_n$ and that $\lim x_n = x$

$= \lim z_n$.

Then $\lim(y_n) = x$.

(b) Show with all details how the Squeeze Theorem can be used to compute $\lim \frac{(-1)^n}{n^2}$.

We know that $\lim \frac{1}{n} = 0$, and that

the product rule implies $\lim \frac{1}{n^2} = 0$

We set $x_n = -\frac{1}{n^2}$, $y_n = (-1)^n \cdot \frac{1}{n^2}$

and $z_n = \frac{1}{n^2}$. Since $\lim \frac{-1}{n^2}$

$= 0 = \lim \frac{1}{n^2}$, it follows that

$\lim \frac{(-1)^n}{n^2} = 0$

9. (a) Use the fact that $\lim(1+\frac{1}{n})^n = e$ to compute $\lim(1+\frac{1}{n^2})^{3n^2}$.

If we set $e_n = (1+\frac{1}{n})^n$, then the subsequence obtained by setting $n = k^2$ is $e_{k^2} = (1+\frac{1}{k^2})^{k^2}$ satisfies

$$\lim (1+\frac{1}{k^2})^{k^2} \to e = \lim (1+\frac{1}{n})^n.$$

and so

$$\lim (1+\frac{1}{k^2})^{3k^2} = e^3.$$

(b) What theorem are you using to compute this limit?

We have used the fact that if (x_n) converges to x, then any subsequence defined by $X' = (x_{n_k})$ also converges to x.