Math 341 Exam 2 Fall 2017 Name ______________

1. Does $\lim_{x \to 0^+} \cos(1/x)$ exist? You must justify your answer.

Set $x_n = \frac{1}{2n\pi}$. Then $\frac{1}{2n\pi} \to 0^+$.

Also $\frac{1}{x_n} = 2n\pi$, so $\cos\left(\frac{1}{x_n}\right) = 2n\pi$, for all $n = 1, 2, \ldots$

Set $y_n = \frac{1}{(2n + \frac{1}{2})\pi}$. Then $\frac{1}{(2n+\frac{1}{2})\pi} \to 0^+$

Also $\frac{1}{y_n} = (2n + \frac{1}{2})\pi$, so $\cos\left(\frac{1}{y_n}\right) = 0$, for all $n = 1, 2, \ldots$

If $\lim_{x \to 0^+} \cos\left(\frac{1}{x}\right)$ exsisted, then

$\cos\left(\frac{1}{x_n}\right)$ and $\cos\left(\frac{1}{y_n}\right)$ would have the same limit.

2. Evaluate $\lim_{x\to\infty} \frac{5+3x}{\sqrt{3+2x}}$. You must justify your answer.

$$\frac{5+3x}{\sqrt{3+2x}} = \frac{x\left(3+\frac{5}{x}\right)}{\sqrt{x}\sqrt{\frac{3}{x}+2}} = \sqrt{x}\,\frac{3+\frac{5}{x}}{\sqrt{2+\frac{3}{x}}}.$$

But $\lim\limits_{x\to\infty} \dfrac{3+\frac{5}{x}}{\sqrt{2+\frac{3}{x}}} = \dfrac{3}{\sqrt{2}}$

Since $x\to\infty$, then $\lim\limits_{x\to\infty}\sqrt{x} = \infty$.

Hence $\lim\limits_{x\to\infty} \sqrt{x}\cdot\dfrac{3+\frac{5}{x}}{\sqrt{2+\frac{3}{x}}} = \infty\cdot\dfrac{3}{\sqrt{2}}$

$$= \infty$$

3. Let

$$f(x) = \begin{cases} x^{3/2} \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Evaluate $f'(0)$. You must justify your answer.

$$\left| \frac{f(x) - 0}{x - 0} \right| = \left| \frac{x^{3/2} \cdot \sin \frac{1}{x^2}}{x} \right|$$

$$\leq \left| \frac{x^{3/2}}{x} \right| = \left| x^{\frac{1}{2}} \right| . \quad \text{Since } \lim_{x \to 0} \left| x^{\frac{1}{2}} \right| = 0$$

it follows that $\lim_{x \to 0} \frac{f(x)}{x} = 0$.

Hence $f'(0) = 0$

4. State the Maximum-Minimum Theorem If f is a continuous function on a closed bounded interval I, then there are numbers x' and x'' in I, so that $f(x') \geq f(x)$ and $f(x'') \leq f(x)$ for all $x \in I$.

5. State the Location of Roots Theorem If f is continuous on a bounded closed interval $[a, b]$ and if $f(a) < 0 < f(b)$, there is an $x_0 \in (a, b)$ so that $f(x_0) = 0$.

6. State the Uniform Continuity Theorem

Theorem: If f is a continuous function on an interval $I = [a, b]$, then f is uniformly continuous. Thus, if $\varepsilon > 0$, then there is a number $\delta > 0$ so that if x', x'' are in I that satisfy $|x' - x''| < \delta$, then $|f(x') - f(x'')| < \varepsilon$.

Thm. If f is a continuous function on a closed bounded interval, then there is an $M > 0$ so that $|f(x)| \leq M$ for all $x \in [a, b]$.

7. State and **prove** the Boundedness Theorem.

Proof: Suppose that the theorem is false.

Then for any $n \in \mathbb{N}$, there is an $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. Since (x_n) is bounded, the Bolzano-Weierstrass Thm. implies that there is a subsequence (x_{n_r}) so that (x_{n_r}) converges to x. Since (x_{n_r}) is bounded above and below by b and a, it must be that $x \in [a, b]$. Since f is continuous at x, it follows that $(f(x_{n_r}))$ is bounded, i.e, $|f(x_{n_r})| \leq M$ for all $r = 1, 2, \ldots$. But $|f(x_{n_r})| \geq n_r \geq r$, so this is a contradiction.

Product Rule. Suppose f and g are both differentiable at c, Then

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

8. State and **prove** the Product Rule for Derivatives.

Note that

$$f(x)g(x) - f(c)g(c)$$

$$= f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)$$

Dividing by $x-c$, we get that the above is

$$\frac{f(x)-f(c)}{x-c} \cdot g(x) + f(c) \cdot \frac{g(x)-g(c)}{x-c}$$

Since $g(x)$ is differentiable, it is continuous at c. We take the limit and obtain

$$f'(c)g(c) + f(c)g'(c) = (fg)'(c).$$

which proves the rule.

9. Show that the function $1/x$ is uniformly continuous on $[1, \infty)$.

If $f(x) = \frac{1}{x}$, then

$$\frac{1}{x'} - \frac{1}{x''} = \frac{x'' - x'}{x' x''}$$

Since $x' \geq 1$ and $x'' \geq 1$, we have $\frac{1}{x'} \leq 1$ and $\frac{1}{x''} \leq 1$.

Hence, $$\left| \frac{1}{x'} - \frac{1}{x''} \right| \leq \frac{|x'' - x'|}{x' x''}$$

$$\leq |x'' - x'|.$$

For any $\varepsilon > 0$, set $\delta = \varepsilon$. Thus if

$|x' - x''| < \delta$, then we have shown that

$\left| \frac{1}{x'} - \frac{1}{x''} \right| \leq |x'' - x'| < \delta = \varepsilon$. This shows $\frac{1}{x}$ is uniformly continuous.

10. Use the Location of Roots Theorem to show that there is a number $c \in (0, \frac{\pi}{2})$ that is a root of the equation $x^2 - \cos x = 0$.

For $\cancel{x} \in [0, \frac{\pi}{2}]$, set $f(x) = x^2 - \cos x$.

Note that $f(0) = 0 - \cos 0 = -1$

and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} - \cos\frac{\pi}{2} = \frac{\pi^2}{4}$.

Since $f(0) < 0 < f(\frac{\pi}{2})$, the Location of Roots Theorem implies there is an $x_0 \in (0, \frac{\pi}{2})$ so that $f(x_0) = 0$, i.e. $x_0^2 - \cos x_0 = 0$