Review Problems for Ex. 1, Fall 2017 Math 341.

1. Suppose that (x_n) is bounded and that $\lim y_n = 0$. Find $\lim x_n y_n$.

2. Suppose that $\lim x_n = x$ and that $x_n \leq 0$. Show that $x \leq 0$.

3. Suppose that $\lim x_n = x$ and $\lim y_n = y$. Show that $\lim x_n y_n = xy$.

4. Show that $\lim y_n = y$ and that $|y| > 0$. Show there is a constant K so that

$$|y_n| > \frac{|y|}{2}, \text{ if } n \geq K.$$

5. Suppose that $\lim y_n = y$ and that $y \neq 0$. Show that

$$\lim \frac{1}{y_n} = \frac{1}{y}.$$

6. Show that $\lim \frac{1}{\sqrt{n}} = 0$

7. Suppose that (x_n) is increasing and bounded above. Show there is an $\tilde{x}$ so that $\lim (x_n) = \tilde{x}$.

8. State and prove Bernoulli's Inequality.

Then show $\lim R^n = \infty$ when $R > 1$.

9. Show that

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{2}\left[n(n+1)\right]^2$$

for all $n \in \mathbb{N}$.

10. Use the formula $\lim \left(1+\frac{1}{n}\right)^n = e$ to prove that $\lim \left(1+\frac{1}{n^2}\right)^{n^2} = e$

11. Suppose that (x_n) is a positive sequence with $\lim x_n = x$, where $x > 0$. Show that $\lim \sqrt{x_n} = \sqrt{x}$.

12. Use the Ratio Test to show that $\lim \frac{3^n}{n!} = 0$.

13. Let $X = (x_n)$ be a sequence of numbers such that $x_n \geq 0$. and such that $\lim x_n = x \geq 0$. Then $\lim (\sqrt{x_n}) = \sqrt{x}$

Pf. There are two cases:

1. $x = 0$, and 2. $x > 0$

We consider Case 2. We can assume that $x_n > 0$.

Note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}$$

$=$

$$= \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Hence $\left|\sqrt{x_n} - \sqrt{x}\right| \le \dfrac{|x_n - x|}{\sqrt{x}}$.

For any $\varepsilon > 0$, choose $K(\varepsilon)$

so that if $n \ge K(\varepsilon)$, then

$|x_n - x| < \sqrt{x} \cdot \varepsilon$, which implies that

$$\left| \sqrt{x_n} - \sqrt{x} \right| < \frac{\sqrt{x} \cdot \varepsilon}{\sqrt{x}} = \varepsilon.$$

Case 1. If $x = 0$, let $\varepsilon > 0$.

Since $x_n \to 0$, there is a number K such that if $n \geq K$, then $0 \leq x_n = x_n - 0 < \varepsilon^2$. Therefore, $0 \leq \sqrt{x_n} < \varepsilon$ for $n \geq K$. Since ε is arbitrary, this implies that $\sqrt{x_n} \to 0$.