MATH 341

An Introduction to Real Analysis

My name is David Catlin

Office is 744 MATH Bldg.

catlind@purdue.edu

My office hours are

MON 2:30

WED 2:30

THURS 11:30

The textbook is

An Introduction to Real Analysis, 4th edition by Bartle and Sherbert

The course home page is

math.purdue.edu/~catlin

Click on MA341 Web Page under Fall Semester 2017

All lectures and homework exercises will be posted online.

In this course we will give a rigorous and detailed study of the ideas and techniques of calculus of one variable, including

1. Set Theory
2. Real Numbers
3. Sequences and Series
4. Limits
5. Continuous Functions

6. Differentiation
7. Integration
8. Sequences and Series of Functions
9. Taylor Series

1.1 Sets and Functions

If x is in a set A, we write

$x \in A$

We also say x is a member of A or that x belongs to A. If x is not in A, we write $x \notin A$.

If every element of a set A belongs to a set B, we say A is a subset of B, and

$A \subseteq B$ or $B \supseteq A$.

Some common sets of numbers are:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ integers

$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$

rational numbers

$\mathbb{R}$: set of real numbers

Sometimes a set A is obtained by specifying a property that determines the elements of A.

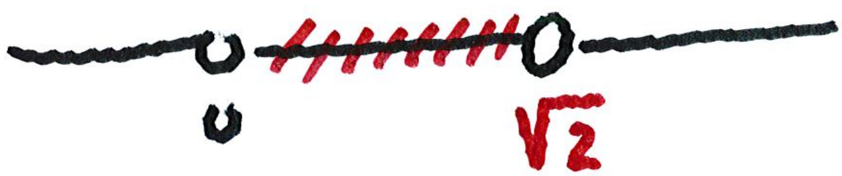
Ex. We say n is an even integer if there is an integer k, so that $n = 2k$.

$$E = \{ n \in \mathbb{Z} : n = 2k, \text{ for any } k \in \mathbb{Z} \}$$

or

$$E = \{ 2k : k \in \mathbb{Z} \}$$

Ex. Let $I = \left\{ x \in \mathbb{Q} : \begin{array}{c} 0 < x \\ \text{and} \\ x^2 < 2 \end{array} \right\}$

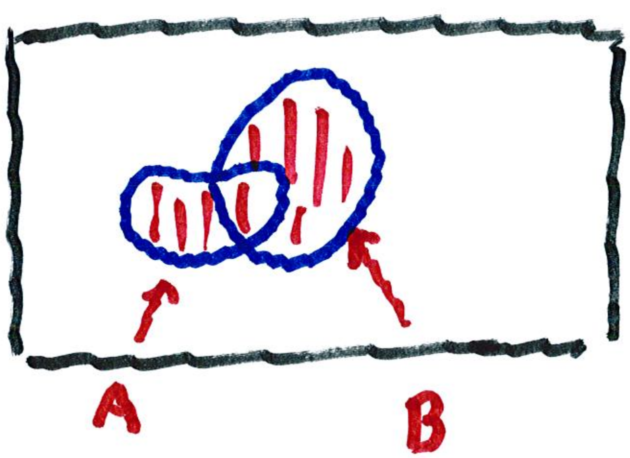


Set Operations

Def (a). The union of sets A and B is

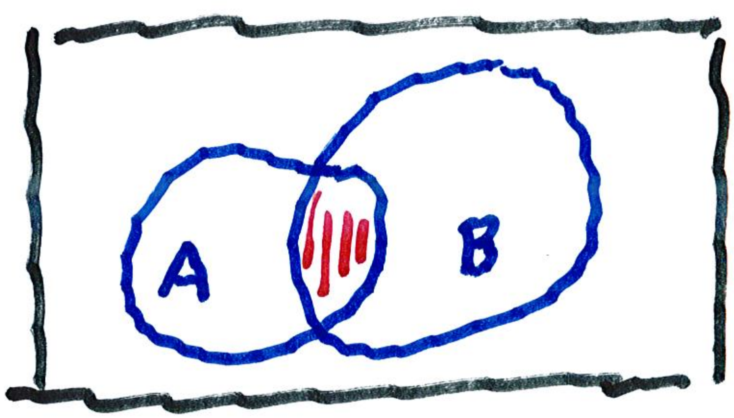
$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

(x can be in both)



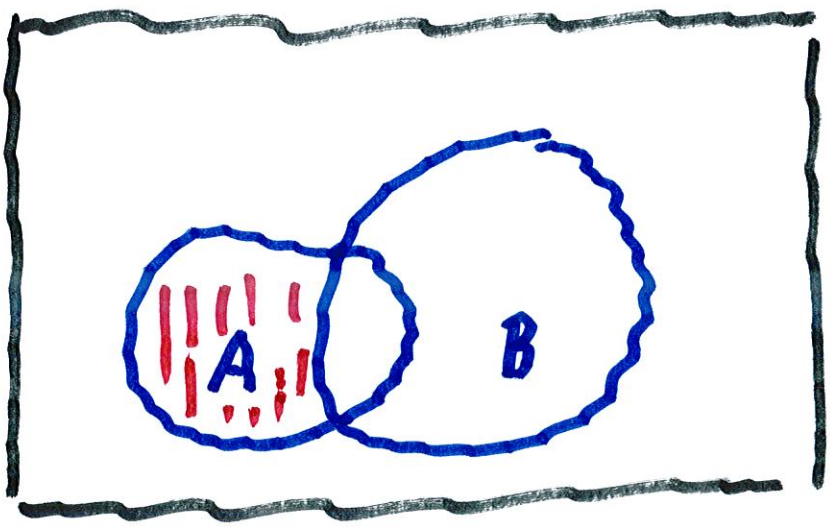
(b) The intersection of the sets A and B is the set

$$A \cap B = \{x \,;\, x \in A \text{ and } x \in B\}$$



(c) The complement of B relative to A is the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$



The set with no elements is the *empty* set, written as $\emptyset$

Two sets A and B are said to be disjoint if there is no element in both A and B.

A and B are disjoint if $A \cap B = \emptyset$

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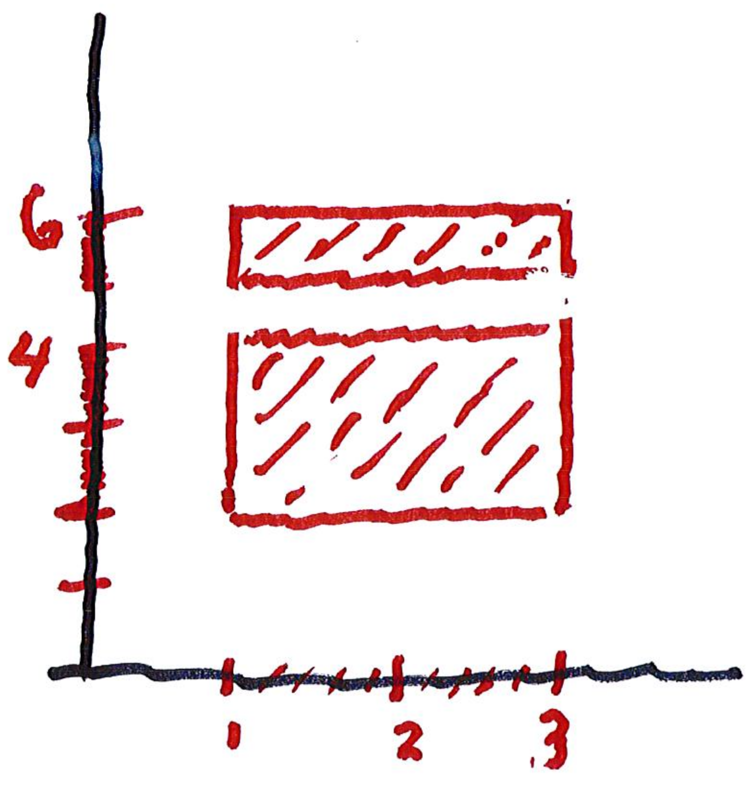
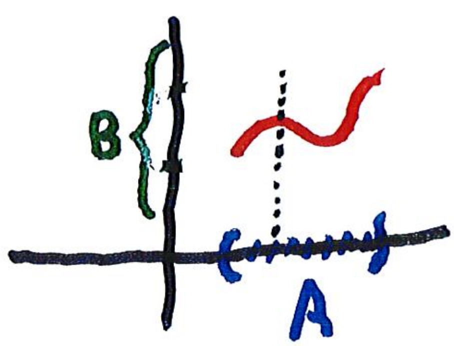
$$A \times B = \left\{ (a,b) : a \in A,\ b \in B \right\}$$

If $A = \left\{ x : 1 \leq x \leq 3 \right\}$

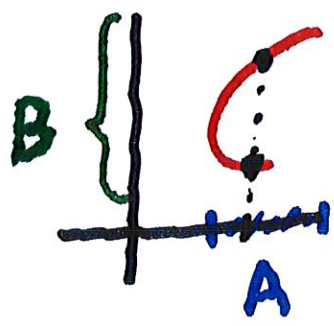
and $B = \left\{ y : 2 \leq y \leq 4 \text{ or } 5 \leq y \leq 6 \right\}$



A function f from A to B is a set f of ordered pairs in $A \times B$ such that for each a in A, there is unique b in B such that $(a, b) \in f$

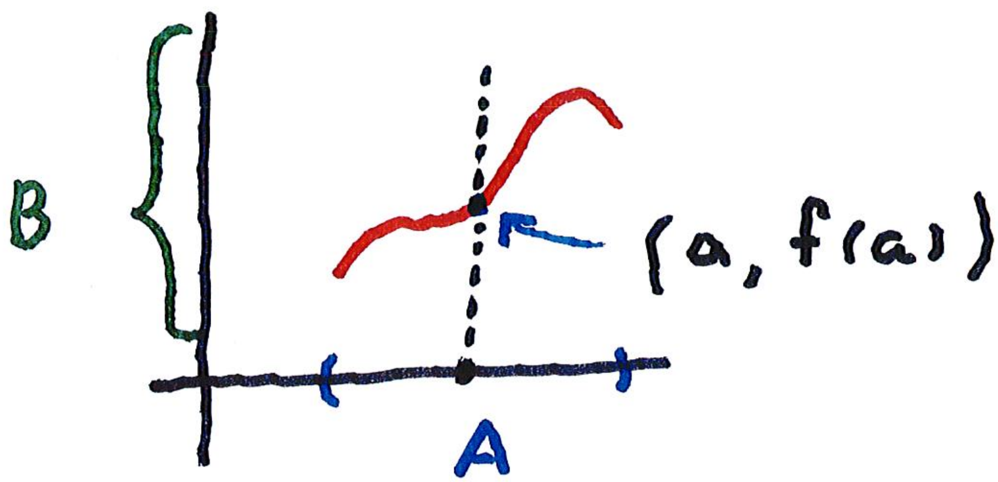


is a fcn.



not a fcn.

If $(a,b) \in f$, we often write $f(a) = b$



We write Domain $= D(f) = A$

Also $R(f) = \left\{ f(a) : a \in A \right\}$

Composition of Fcns.

If A, B, and C are maps,

and $f: A \to B$ and $g: B \to C$

then the composition of f and g is

$$(g \circ f)(x) = g(f(x))$$

for all x in A

Ex. Suppose $f(x) = x^4 - 1$ for x in $(-\infty, \infty)$

and $g(x) = \sqrt{x}$, for

$0 \leq x < \infty$,

then we cannot form

$$(g \circ f)(x) = \sqrt{x^4 - 1}.$$

The problem is $x^4 - 1 < 0$

if $-1 < x < 1$,

because $\sqrt{x^4 - 1}$ only makes sense

if $x^4 - 1 \geq 0$, i.e., if $|x| \geq 1$.

Then we modify f by

defining $f(x) = x^4 - 1$ for $|x| \geq 1$.

Definition. A function

$f : A \to B$ is injective,

if whenever $x_1 \neq x_2$,

then $f(x_1) \neq f(x_2)$. (f is 1-to-1)

Equivalently, if whenever

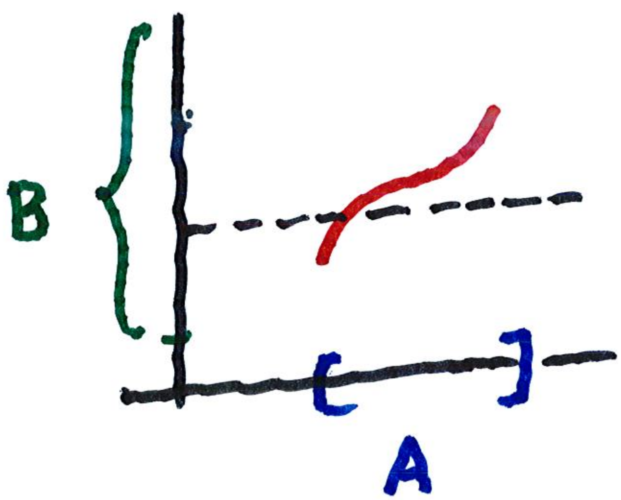
$f(x_1) = f(x_2)$, then $x_1 = x_2$.

Also $f : A \to B$ is surjective

if whenever $y \in B$, then

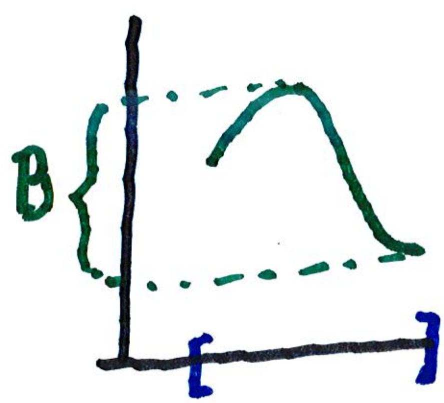
there is an x in A so $f(x) = y$

{ f is onto }



f is 1-to-1

but not onto



f is onto

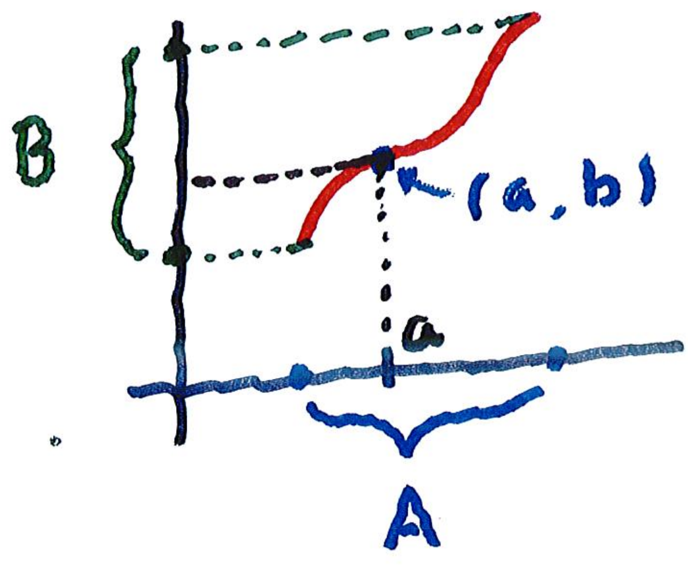
but not

1-to-1

We say f is bijective

if f is both injective

and surjective.



$f(a) = b$

$g(b) = a.$

Theorem. Suppose $f: A \longrightarrow B$

is bijective, (i.e., both onto and 1-to-1)

Then there is a bijection

$g: B \longrightarrow A$ that satisfies

(a) $g(f(a)) = a$ for all a in A

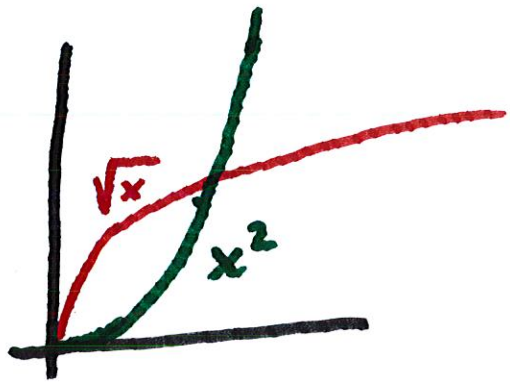
(b) $f(g(b)) = b$ for all b in B

We write $g = f^{-1}$ and $f = g^{-1}$

The formula in (a) shows that g is surjective. For any a in A, $f(a)$ is the value of x such that $g(x) = a$.

(b) shows that g is injective. For if $g(b_1) = g(b_2)$. Then $f(g(b_1)) = f(g(b_2))$, so that $b_1 = b_2$.

Ex. Let $f(x) = x^2$. The
$(0 \le x < \infty)$
inverse
of f is $\sqrt{x}$.



$x^2 = 3$

Apply $\sqrt{}$. $\sqrt{x^2} = \sqrt{3}$

or $x = \sqrt{3}$.

Ex. $\sin x$ maps $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1,1]$

$\sin^{-1}$ maps $[-1, 1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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Suppose $\sin x = .42$

Apply $\sin^{-1}$:

$$\sin^{-1}(\sin x) = \sin^{-1}(.42)$$

$$\to \quad x = \sin^{-1}(.42)$$

Ex. Let $A = \{x \in \mathbb{R} : x \neq -1\}$

and let $f(x) = \dfrac{2x-1}{x+1}$.

Show that f is injective.

Suppose $f(x_1) = f(x_2)$

$$\frac{2x_1 - 1}{x_1 + 1} = \frac{2x_2 - 1}{x_2 + 1}$$

$$(2x_1 - 1)(x_2 + 1) = (2x_2 - 1)(x_1 + 1)$$

$$2x_1 - x_2 = -x_1 + 2x_2$$

$$\rightarrow \quad 3x_1 = 3x_2$$

or $x_1 = x_2$. ✓

Now find the range of f.

Find all y, such that

$$y = \frac{2x-1}{x+1} \rightarrow yx + y = 2x - 1$$

Solve for x: $(y-2)x = -y-1$

$$\rightarrow \quad x = \frac{y+1}{2-y}$$

This can be solved only

if $y \neq 2$. $R(f) = \{ y \in \mathbb{R} : y \neq 2 \}$