3.2 Limit Thms.

Given 2 sequences

$X = (x_n)$ and $Y = (y_n)$ such that

$\lim (x_n) = x$ and $\lim (y_n) = y$,

we proved that

1. $\lim (x_n + y_n) = x + y$

2. $\lim (x_n y_n) = xy$.

3 To prove $\lim (c x_n) = cx$,

let $Y = (y_n) = c$, for all c.

Then $\lim cx_n = \lim y_n x^n$

$= \lim y_n \cdot \lim x_n$

$= cx$ $\therefore \lim cx_n = cx.$

---

4. Now suppose $Z = (z_n)$ and

that $\lim (z_n) = z \neq 0.$

Choose $K_1 \in \mathbb{N}$ so that if $n \geq K_1$,

then $|z_n - z| < \frac{|z|}{2}.$

It follows that

$$
\begin{aligned}
|z_n| &= |(z_n - z) + z| \\
&= |z + (z_n - z)| \\
&\geq |z| - |z_n - z| \\
&\geq |z| - \frac{|z|}{2} = \frac{|z|}{2}.
\end{aligned}
$$

We use this to estimate the limit of $\frac{1}{z}$:

$$
\left| \frac{1}{z_n} - \frac{1}{z} \right| = \left| \frac{z - z_n}{z_n z} \right|
$$

$$\le |z - z_n| \cdot \frac{2}{|z|^2}$$

since $\dfrac{1}{|z_n|} \le \dfrac{2}{|z|}$ when

$n \ge K_1$. Now choose $\varepsilon > 0$

and choose K_2 so that

$|z_n - z| < \dfrac{|z|^2}{2}\varepsilon$ when $n \ge K_2$.

Now set $K = \mathrm{Max}\{K_1, K_2\}$.

If $n \ge K$, then

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| \leq |z - z_n| \cdot \frac{2}{|z|^2}$$

$$< \frac{|z|^2}{2} \cdot \varepsilon \cdot \frac{2}{|z|^2} = \varepsilon$$

This shows that $\lim \left(\frac{1}{z_n} \right) = \frac{1}{z}$.

Ex. Use the Limit Laws to compute $\lim \dfrac{n^2 + 2n}{3n^2 + 1}$

$$\frac{n^2 + 2n}{3n^2 + 1} = \frac{n^2 \left(1 + \frac{2}{n} \right)}{n^2 \left(3 + \frac{1}{n^2} \right)}$$

$$= \frac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}} .$$

Since $\lim \frac{1}{n} = 0$,

we have $\lim \frac{2}{n} = 0$ and $\lim \frac{1}{n^2} = 0$

$\therefore \lim (1 + \frac{2}{n}) = 1 + 0 = 1$

and $\lim (3 + \frac{1}{n^2}) = 3$

Hence the Quotient Rule

$\rightarrow \lim \dfrac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}} = \dfrac{1}{3}$

To show that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$,

we first show:

If $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$

Suppose that $\sqrt{a} \geq \sqrt{b}$.

Then $a = \sqrt{a}\sqrt{a} \geq \sqrt{b}\sqrt{a} \geq \sqrt{b}\sqrt{b} = b$.

This contradicts the hypothesis

that $a < b$.

We now can prove:

$$\lim \frac{1}{\sqrt{n}} = 0$$

Proof: Choose $\varepsilon > 0$. Then choose an integer K so that $K > \frac{1}{\varepsilon^2}$. If $n \in \mathbb{N}$ and $n \geq K$, then $n \geq K > \frac{1}{\varepsilon^2}$. This gives

$$\sqrt{n} > \sqrt{\frac{1}{\varepsilon^2}} = \frac{1}{\varepsilon},$$

which gives

$$\frac{1}{\sqrt{n}} < \varepsilon,$$

which implies

$$\left| \frac{1}{\sqrt{n}} - 0 \right|$$

if $n \geq K$. We conclude that $\lim \frac{1}{\sqrt{n}} = 0$.

Ex. Compute $\lim \frac{\sqrt{n}}{2n+3}$

Factor out highest power

$$= \frac{\sqrt{n} \cdot 1}{n\left(2+\frac{3}{n}\right)} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\left(2+\frac{3}{n}\right)}$$

Note $\lim \frac{1}{\sqrt{n}} = 0$ and

$\lim \frac{1}{2+\frac{3}{n}} = \frac{1}{2}$.

$\therefore$ Product Rule implies

$$\lim \frac{\sqrt{n}}{2n+3} = 0 \cdot \frac{1}{2} = 0$$

Thm. Suppose $\lim x_n = x$ and that $x_n \leq 0$. Then $x \leq 0$.

Pf. Suppose statement is not true, i.e., suppose $x > 0$.

Pick $\varepsilon = x$

Then there is K, so if $n \geq K$, then $|x_n - x| < x$

Hence $-\varepsilon < x_n - x < \varepsilon$.

$-x < x_n - x \rightarrow 0 < x_n$.

This contradicts hypothesis that $x_n \leq 0$

Corollary. Suppose (x_n) and (y_n) are both convergent and that $x_n \leq y_n$, all n.

Then $x \leq y$

Pf. Set $z_n = x_n - y_n$.

Then $z_n \le 0$, for all n.

Hence the theorem implies

$\lim z_n = z$, i.e., $z \le 0$.

$\therefore$ $x - y \le 0$.

i.e. $\lim(x_n) \le \lim(y_n)$.

Suppose $a \le x_n \le b$ and

that (x_n) is convergent.

Then $a \le \lim(x_n) \le b$.

Pf. To prove $\lim(x_n) \le b$,

set $(y_n) = (b)$ for all n.

The hypothesis that $x_n \le b$

(using previous result)

implies that $\lim(x_n) \le \lim(y_n)$,

or $\lim(x_n) \le b$.

Similarly, if we set $y_n = a$,

for all n, then the

hypothesis $\Rightarrow$ $y_n \le x_n$,

which implies $a \leq \lim(x_n)$.

Squeeze Thm.

Suppose that $X = (x_n)$,

$Y = (y_n)$, and $Z = (z_n)$ are

sequences with

$$x_n \leq y_n \leq z_n.$$

Suppose also that $\lim(x_n) = \lim(z_n)$

Then $\lim(x_n) = \lim(y_n) = \lim(z_n)$

Proof: Let $w = \lim(x_n)$

$= \lim(z_n)$.

For any $\varepsilon > 0$, choose K so

that if $n \geq K$, then

$|x_n - w| < \varepsilon$ and $|z_n - w| < \varepsilon$.

$\rightarrow -\varepsilon < x_n - w \leq y_n - w \leq z_n - w < \varepsilon$

$\rightarrow -\varepsilon < y_n - w < \varepsilon$

$\rightarrow \lim y_n = w$

Ex. Compute $\lim \frac{(-1)^n}{n}$.

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}.$$

Also, we know that

$\lim \frac{1}{n} = 0$ and $\lim \frac{-1}{n}$.

$\therefore$ Squeeze Thm. $\rightarrow \lim \frac{(-1)^n}{n} = 0$

<u>Ratio Test</u> for Sequences:

Let (x_n) be a sequence of positive numbers such that

$L = \lim \left\{ x_{n+1}/x_n \right\}$ exists.

If $L < 1$, then $\lim(x_n) = 0$.

Let r be a number satisfying

$L < r < 1$ and Let $\varepsilon = r - L$.

There is a number K so that

if $n \geq K$, then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon.$$

It follows that

$$\frac{x_{n+1}}{x_n} - L < \varepsilon = r - L$$

$$\therefore \quad \frac{x_{n+1}}{x_n} < r \quad \text{for all } n \geq K.$$

Hence $0 < x_{n+1} < r x_n$ for all $n \geq K$.

Then $x_{k+1} < r x_k$

$$x_{k+2} < r x_{k+1} < r^2 x_k$$

$$x_{k+3} < r^3 x_k$$

$$\vdots$$

$$x_{k+n} < r^n$$

Since $\lim r^n = 0$, it follows

that $\lim x_{k+n} = 0$,

which implies $\lim x_n = 0$

Ex. Show that $\lim \underbrace{n^k \rho^n}_{} = 0$

$0 < \rho < 1$,

k a positive integer.

Look at

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^k \rho^{n+1}}{n^k \rho^n}$$

$$= \left(\frac{n+1}{n}\right)^k \rho$$

$$= \left(1 + \frac{1}{n}\right)^k \rho.$$

Since $1 + \frac{1}{n} \to 1$, we obtain

$\lim \frac{a_{n+1}}{a_n} = \rho$. $\therefore$ Ratio Test

$\to \lim_{n \to \infty} a_n = 0$