which implies $a \le \lim(x_n)$.

Squeeze Thm.

Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences with

$$x_n \le y_n \le z_n.$$

Suppose also that $\lim(x_n) = \lim(z_n)$

Then $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

Proof: Let $w = \lim (x_n)$

$= \lim (z_n)$.

For any $\varepsilon > 0$, choose K so that if $n \geq K$, then

$|x_n - w| < \varepsilon$ and $|z_n - w| < \varepsilon$.

$\Rightarrow -\varepsilon < x_n - w \leq y_n - w \leq z_n - w < \varepsilon$

$\rightarrow -\varepsilon < y_n - w < \varepsilon$

$\rightarrow \lim y_n = w$

Ex. Compute $\lim \frac{(-1)^n}{n}$.

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}.$$

Also, we know that

$\lim \frac{1}{n} = 0$ and $\lim \frac{-1}{n}$.

$\therefore$ Squeeze Thm. $\rightarrow \lim \frac{(-1)^n}{n} = 0$

Ratio Test for Sequences:

Let (x_n) be a sequence of positive numbers such that

Ratio Test for Sequences.

Let (x_n) be a sequence of positive numbers such that

$$L = \lim\left(\frac{x_{n+1}}{x_n}\right) \text{ exists.}$$

If $L < 1$, then $\lim x_n = 0$.

Let r be a number satisfying $L < r < 1$, and let $\varepsilon = r - L$.

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon .$$

It follows that

$$\frac{x_{n+1}}{x_n} - L < \varepsilon = r - L .$$

Hence, $\dfrac{x_{n+1}}{x_n} < r$, for all $n \geq K$.

Therefore

$$0 < x_{n+1} < r x_n \quad \text{for all } n \geq K.$$

Then $x_{K+1} < r x_K$

$$x_{K+2} < r x_{K+1} < r^2 x_K$$

$$x_{K+3} < r^3 x_K$$

$$\vdots$$

$$x_{K+n} < r^n x_K, \quad n = 1, 2, \ldots$$

Now replace n by $n-K$.

$$x_n < r^{-K} x_K r^n,$$

for $n = 1, 2, \ldots$

Since $\lim r^n = 0$, it follows that $\lim x_n = 0$. In fact, if $\varepsilon > 0$ and if $C = r^{-K} x_K$,

then we just choose A

sufficiently large so that

$$r^n < \frac{\varepsilon}{C} \quad \text{if } n \geq A.$$

Thus $C r^n < C \cdot \frac{\varepsilon}{C} = \varepsilon$,

so we conclude that

$$\lim x_n = 0.$$

3.3 Monotone Sequences

Definition. We say a sequence (x_n) is increasing if

$$x_n \le x_{n+1}, \qquad \text{all } n = 1, 2, \ldots$$

That is

$$x_1 \le x_2 \le \ldots x_n \le x_{n+1} \le \ldots$$

We say (y_n) is decreasing if

$$y_n \ge y_{n+1}, \qquad n = 1, 2, \ldots$$

That is

$$y_1 \ge y_2 \ge \ldots \ge y_n \ge y_{n+1} \ge \ldots$$

If (x_n) is increasing or decreasing, we say (x_n) is monotone.

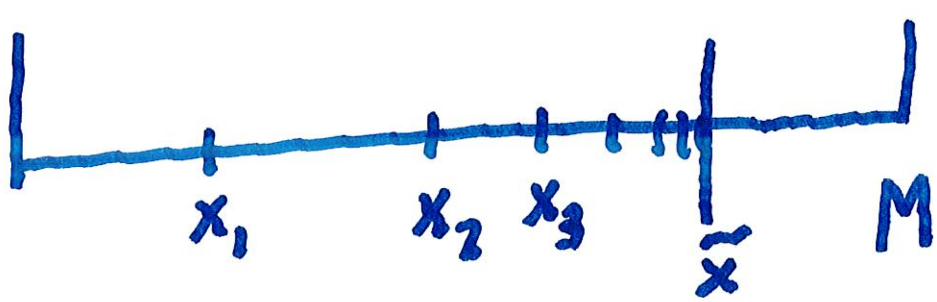
Monotone Convergence Thm.

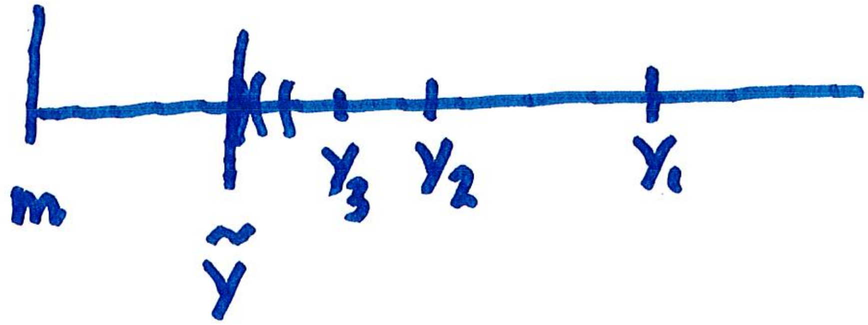
If (x_n) is a bounded monotone sequence, then it converges. In fact, if (x_n) is increasing and bounded, then

$$\lim(x_n) = \tilde{x} = \sup\{x_n : n \in N\}$$

Also, if (y_n) is decreasing and bounded, then

$$\lim(y_n) = \tilde{y} = \inf\{y_n : n \in N\}$$

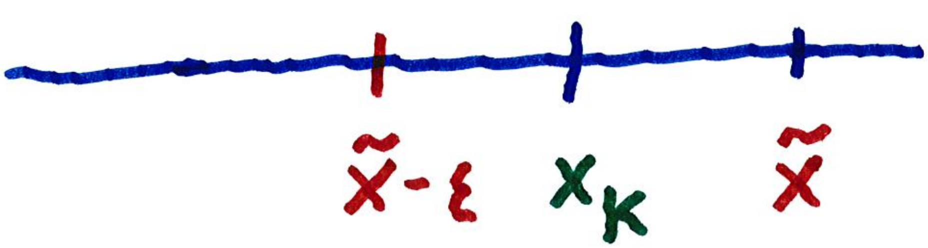




Proof. Since $x_n \le M$ for all $n \in N$, we define

$$\tilde{x} = \sup\{x_n : n \in N\}.$$

For any $\varepsilon > 0$, $\tilde{x} - \varepsilon$ is not an upper bound. It follows that there is a $K \in N$, such that $\tilde{x} - \varepsilon < x_K \le \tilde{x}$.



Since (x_n) is increasing,

if $n \geq K$, then

$$\tilde{x} - \varepsilon < x_K \leq x_n \leq \tilde{x} < \tilde{x} + \varepsilon,$$

where the inequality

$x_n \leq \tilde{x}$ comes from the fact

that $\tilde{x}$ is an upper bound

of $\{x_n : n \in N\}$

It follows that $|x_n - \tilde{x}| < \varepsilon$

if $n \geq K$. Hence $\lim(x_n) = \tilde{x}$.

In the case of (y_n),

$y_n \geq m$ for all n, which

implies that there is a

number $\tilde{y} = \inf\{y_n : n \in N\}$

For any $\varepsilon > 0$, there is a K'

so that $\tilde{y} \leq y_K < \tilde{y} + \varepsilon$.

Since (y_n) is decreasing,

we obtain that if $n \geq K'$, then

$$\tilde{y} + \varepsilon > y_{K'} \geq y_n \geq \tilde{y} > \tilde{y} - \varepsilon,$$

or that

$$\tilde{y} - \varepsilon < y_n < \tilde{y} + \varepsilon, \text{ for } n \geq K'$$

It follows that $\lim (y_n) = \tilde{y}$,

which proves the theorem.

We now use the Least Upper Bound Property to evaluate the lim of some sequences.

Ex. Let $0 < r < 1$. Then

$$\lim r^n = 0.$$

Note that (r^n) is decreasing. Since $r^n > 0$, it follows

that $\underline{\lim r^n = R}$, where $R \geq 0$.

In fact, for any ε, there is a K, so that if $n \geq K$, then

$|r^n - R| < \varepsilon.$

Since $n+1 \geq K$, it follows

that $r^{n+1} \geq K$. Hence,

$|r^{n+1} - R| < \varepsilon$, which

implies that $\lim r^{n+1} = R$.

On the other hand,

$$\lim (r^{n+1}) = \lim (r^n \cdot r)$$

$$= R \cdot r$$

Since $R = Rr$, it follows

that $\underline{\underline{R = 0}}$. Thus, we have

proved that $\lim r^n = 0$.

Ex. Define $y_{n+1} = \frac{2}{5} y_n + 1$,

with $y_1 = 1$.

Assume first

that $\lim y_n = y$. Then we have

$$y = \frac{2}{5} y + 1 \to \frac{3}{5} y = 1$$

$$\to \underline{y = \frac{5}{3}}.$$

Use induction to show that

if $1 \le y_n \le 4$, then y_{n+1} also satisfies $1 \le y_{n+1} \le 4$.

In fact, if $y_n \ge 1$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \ge \frac{2}{5} \cdot 1 + 1 > 1$$

Similarly, if $y_n \le 4$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \le \frac{8}{5} + 1 \le \frac{13}{5}.$$

Now we show that $\underline{y_{n+1} > y_n}$

This is obvious when $n=1$.

Now assume that $y_{n+1} > y_n$.

Then $\frac{2}{5} y_{n+1} > \frac{2}{5} y_n$,

which gives

$$\frac{2}{5} y_{n+1} + 1 > \frac{2}{5} y_n + 1$$

or $\underline{y_{n+2} > y_{n+1}}$.

Since $y_n \leq 4$ for all $n \in \mathbb{N}$,

and since (y_n) is increasing,

we conclude that there

is a $y \in [1, 4]$ such that

$\lim(y_n) = y$ and $\lim(y_{n+1}) = y$.

This implies that

$y = \frac{2}{5}y + 1 \rightarrow y = \frac{5}{3}$.

Ex. Study the convergence of

$$Y_n = \left\{ \frac{1}{n+1} + \cdots + \frac{1}{2n} \right\}.$$

Note that

$$Y_{n+1} = -\frac{1}{n+1} + \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \right\}$$

$$\rightarrow Y_{n+1} = Y_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= Y_n + \frac{1}{(2n+1)(2n+2)}$$

$$\therefore \; Y_{n+1} > Y_n$$

Note also that

$$y_n \le n \cdot \frac{1}{n+1} \le 1 \text{ for all } n.$$

Hence the Monotone Convergence Theorem implies that

$$y_n \to y \le 1 \text{ as } n \to \infty.$$

HW # 1. p. 77.

Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2} x_n + 2$.

Show that (x_n) is bounded

and decreasing. Find the limit.

$$x_{n+1} = \frac{1}{2} x_n + 2$$

$x_n = \frac{1}{2} x_{n-1} + 2.$ Subtracting:

$$x_{n+1} - x_n = \frac{1}{2} (x_n - x_{n-1}). \qquad (1)$$

$\therefore$ Monotone Conv. Thm implies

$(x_n) \to s$, some $s > 0$.

$\therefore$ $s = \frac{1}{2}s + 2$. $\Rightarrow s = 4$

HW #2, Let $x_1 > 1$ and

$x_{n+1} = 2 - \frac{1}{x_n}$. Show (x_n) is

monotone and bounded.