4.1 Limits of Functions.

Let $A \subseteq \mathbb{R}$. A point c in $\mathbb{R}$ is is a cluster point of A if for every $\delta > 0$, there is at least one point $x \in A$, $x \neq c$, such that $|x-c| < \delta$.

One can also say c is a cluster pt. of A if every δ-neighborhood $V_\delta(c) = (c-\delta, c+\delta)$ of c contains at least one point of A distinct from c.

Thm. A number c in $\mathbb{R}$ is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

If c is a cluster point of A, then for any $n \in \mathbb{N}$, the $(1/n)$-neighborhood $V_{1/n}(c)$ contains at least one point a_n in A distinct from c.

Then $a_n \in A$, $a_n \neq c$ and

$|a_n - c| < \frac{1}{n}$ implies $\lim(a_n) = c$.

Verify converse on p. 104

Examples.

1. If $A = (0,1)$, then $c=0$ and $c=1$ are also cluster points as well as all points in $(0,1)$.
2. A finite set A has no cluster points.

3. $A = \left\{ \frac{1}{n} : n \in N \right\}$ has only the point 0 as a cluster pt.

4. If $A = Q$, the set of rational points, then every point in $\mathbb{R}$ is a cluster point of A.

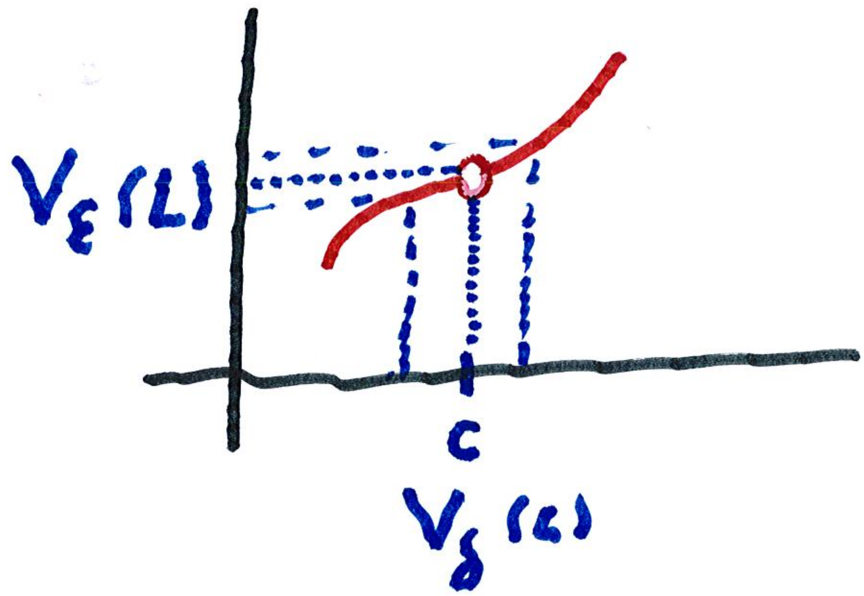
The main idea about cluster points is that one defines limits of functions at such points

Definition of the Limit

Definition. Let $A \subset \mathbb{R}$ and let c be a cluster point of A. For a function $f: A \to \mathbb{R}$, a number L is said to be a limit of f at c if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

We say f converges to L at c,

and we write $L = \lim_{x \to c} f(x)$.



Thm. If $f: A \longrightarrow \mathbb{R}$ and if c is a cluster point of A, then f can only have one limit at c.

Pf. Suppose that

$$\lim_{x\to c} f = L_1 \quad \text{and} \quad \lim_{x\to c} f = L_2 .$$

Assuming $L_1 \neq L_2$, set $\varepsilon = \dfrac{|L_1 - L_2|}{2}$,

and choose δ_1 and $\delta_2 > 0$

so that if $0 < |x-c| < \delta_1$ and

if $0 < |x-c| < \delta_2$, then

$|f(x) - L_1| < \varepsilon$ and

$|f(x) - L_2| < \varepsilon$, respectively.

Setting $\delta = \min\{\delta_1, \delta_2\}$, and

if $0 < |x-c| < \delta$, then

$$|L_1 - L_2| = |(L_1 - f(x)) - (L_2 - f(x))|$$

$$\leq |L_1 - f(x)| + |L_2 - f(x)|$$

$$< \varepsilon + \varepsilon$$

$$= \frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2}$$

$$= |L_1 - L_2|.$$

This contradiction implies that $L_1 = L_2$.

Show that if $h(x) = x^2$, then

$$\lim_{x \to c} x^2 = c^2.$$

Note that

$$|x^2 - c^2| = |x+c| \cdot |x-c| .$$

We estimate $|x+c|$:

$$|x+c| = |(x-c) + 2c|$$

$$\leq 1 + 2|c|, \text{ if } |x-c| < 1.$$

Now, for a given $\varepsilon > 0$, set

$$\delta(\varepsilon) = \min\left\{1, \frac{\varepsilon}{1+2|c|}\right\}$$

Hence, if $0 < |x-c| < \delta(\varepsilon)$, then

$$|x+c||x-c| < (2|c|+1) \cdot \frac{\varepsilon}{1+2|c|}$$

$$= \varepsilon.$$

Hence, $\lim_{x \to c} x^2 = c^2$.

Ex. Show that $\lim_{x \to 2} \frac{x^2 - 3x}{x+3} = \frac{-2}{5}$.

Let $\psi(x) = \frac{x^2 - 3x}{x+3}$. Then

$$\left|\psi(x) + \frac{2}{5}\right| = \left|\frac{5x^2 - 15x + 2(x+3)}{5(x+3)}\right|$$

$$= \frac{|5x^2 - 13x + 6|}{5|x+3|}$$

$$= \frac{|5x-3|}{5|x+3|}\,|x-2|$$

Note that if $|x-2| \le 1$, then

$1 \le x \le 3$. Hence, if $|x-2| \le 1$,

$|5x-3| \le |5x| - 3 \le 12$

and $5|x+3| \ge 5 \cdot 4 = 20$, which

implies that $\dfrac{|5x-3|}{5|x+3|} \le \dfrac{12}{20}|x-2|$

For a given $\varepsilon > 0$, set

$$\delta(\varepsilon) = \min\left\{1, \frac{5\varepsilon}{3}\right\}$$

If $|x-2| < \delta(\varepsilon)$, then

$$\left|\psi(x) - \left(-\frac{2}{5}\right)\right| < \varepsilon.$$

The following makes it possible to convert function limits into corresponding questions about sequence limits.

Thm. Let $f: A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent:

(i) $\lim\limits_{x \to c} f = L$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $\big(f(x_n)\big)$ converges to L.

Proof. (i) $\Rightarrow$ (ii). Assume that f has limit L at c, and suppose (x_n) is a sequence in A with $\lim(x_n) = c$ and $x_n \neq c$ for all n. We must prove that the sequence $(f(x_n))$ converges to L.

Let $\varepsilon > 0$ be given. Then by definition of function limits there exists $\delta > 0$ such that if $x \in A$ satisfies $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$.

Since (x_n) converges to c, for a given $\delta > 0$, there exists a number $K(\delta)$ such that if $n > K(\delta)$, then $|x_n - c| < \delta$.

But for each such x_n, we have $|f(x_n) - L| < \varepsilon$.

Now we prove (ii) $\Rightarrow$ (i).

We argue by contradiction.

If (i) is not true, then

there exists an ε_0-neighborhood

$V_{\varepsilon_0}(L)$ such that no matter which

δ-neighborhood of c we pick,

there will be at least one

number x_δ in $A \cap V_\delta(c)$ with

$x_\delta \neq c$ such that $f(x_\delta) \notin V_{\varepsilon_0}(L)$.

Hence, for every $n \in \mathbb{N}$,

the $(1/n)$-neighborhood of c

contains a number x_n such that

$0 < |x_n - c| < \delta$ and $x_n \in A$,

but such that

$|f(x_n) - L| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

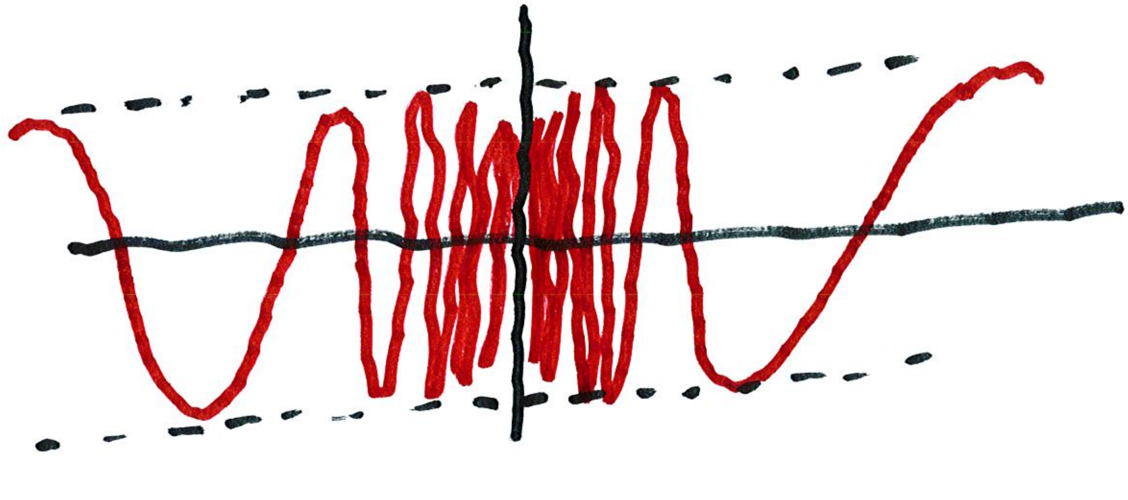
We've shown that the sequence

(x_n) in $A \setminus \{c\}$ converges to c

but $(f(x_n))$ does not converge to L. Thus, we've shown (ii) is NOT true. This contradiction implies that (ii) implies (i).

Divergence Criterion. The function f does not have a limit at c if and only if there is a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c, but the sequence $(f(x_n))$ does NOT converge.

Ex. $\lim_{x \to 0} \sin(1/x)$ does not exist.



Set $x_n = \dfrac{1}{n\pi + \frac{\pi}{2}}$

$$\sin\left(\frac{1}{x_n}\right) = \sin\left(n\pi + \frac{\pi}{2}\right)$$

If n is even, then

$$\sin\left(n\pi + \frac{\pi}{2}\right) = 1.$$

If n is odd, then

$$\sin\left(n\pi + \frac{\pi}{2}\right) = -1.$$

$\therefore$ $x_n \to 0$, and $x_n \neq 0$, but

$\sin(1/x_n)$ does not converge.

$\Rightarrow$ $\sin(1/x)$ has no limit at $x=0$