

Principle of 1.2 Mathematical Induction

Let S be a subset of N

that satisfies

(1) The number $1 \in S$

(2) For every $k \in N$,

if $k \in S$, then $k+1 \in S$

Then for all $n \in N$, $n \in S$.

Note (2) does not ask
us to prove that $k \in S$.

We only need to show

that "if $k \in S$, then $k+1 \in S$ ".

Usually, Math. Ind. is used to prove that a sequence of statements are all true.

For each $n \in \mathbb{N}$, let $P(n)$ be a meaningful statement about $n \in \mathbb{N}$. We let

$$S = \{n \in \mathbb{N}; P(n) \text{ is true.}\}$$

The above Mathematical

Induction Principle becomes:

Suppose that

(1) $P(1)$ is true.

(2') For every $k \in \mathbb{N}$, if

$P(k)$ is true, then

$P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Ex. Suppose $P(n)$ is the statement that

$$f(n) = n^2 - n + 41 \text{ is prime.}$$

Note that when $n=1$,

$$1^2 - 1 + 41 \text{ is prime.}$$

Then $P(1)$ is true.

But (after some calculation)

$$f(40) = 1601 \text{ is prime and}$$

$$f(41) = 41^2 = 1681$$

$\therefore f(41)$ is NOT prime.

Hence $P(40)$ is true but

$P(41)$ is false.

Thus (2) fails when $n = 40$

Ex. Use Math. Ind. to prove

that

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

When $n = 1$, $P(1)$ is the
statement

$$1^2 = \frac{1 \cdot 2 \cdot (3)}{6} = 1$$

$\therefore P(1)$ holds.

Now suppose $P(k)$ is true.

Then

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

by the induction
assumption

Now check $P(k+1)$:

$$1^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right]$$

$$= \frac{(k+1)}{6} \left[k(2k+1) + 6k + 6 \right]$$

$$= \frac{(k+1)}{6} \left[2k^2 + 7k + 6 \right]$$

$$= \frac{(k+1)}{6} (k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$\therefore P(k+1)$ is true.

$\therefore (2)$ holds

Since both (1) and (2) are true, it follows that

$P(n)$ is true for all $n \in \mathbb{N}$.

Hence

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Ex. Prove that $5^{2^n} - 1$ is
divisible by 8. \curvearrowright This
is $P(n)$.

(1) when $n=1$,

$$5^2 - 1 = 24 = 3 \cdot 8,$$

so $P(1)$ is true.

(2) Suppose that $P(k)$ is true,

i.e., $5^{2^k} - 1$ is divisible
by 8.

Check $P(k+1)$:

$$5^{2(k+1)} - 1 = 5^2 \cdot 5^{2k} - 1$$

$$= 5^2 (5^{2k} - 1) + 5^2 - 1$$

$$= 5^2 (5^{2k} - 1) + (5^2 - 1)$$

↑
↑

Both are divisible by 8

$\therefore P(k+1)$ is true.

\rightarrow (2) holds $\Rightarrow 5^{2n} - 1$ is div. by 8 for all n .

Bernoulli's Inequality

Show that

for all $n \in \mathbb{N}$ and for all $x > -1$,

$$(1+x)^n \geq (1+nx)$$

Pf. First we check $P(1)$

$$(1+x)^1 = (1+1 \cdot x) \quad \checkmark.$$

Now check (2)

Suppose that $(1+x)^k \geq 1+kx$

for all $x > -1$.

Note that

$$(1+x)^{(k+1)} = (1+x)^k (1+x)$$

$$\geq (1+kx)(1+x)$$

by the inductive hypothesis

and that $1+x > 0$.

$$= 1 + kx + x + kx^2$$

$$\geq 1 + (k+1)x.$$

Thus $P(k+1)$ is true,

and hence (2) holds.

By induction, $P(n)$ is true

for all $n \in \mathbb{N}$

$$\Rightarrow (1+x)^n \geq 1+nx, \quad \text{when } x > -1.$$

Sometimes, the statement
is only defined for $n \geq n_0$

Modified Principle of

Math. Induction.

Suppose that

(1) $P(n_0)$ is true.

(2) For all $k \geq n_0$, if $P(k)$ is
true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

Ex. Prove that

$$2^n < n! \text{ for all } n \geq 4.$$

Note that when $n = 4$,

$$2^4 = 16 < 24 = 4!$$

This shows that $P(4)$ holds.

Now let k be an integer

≥ 4 , and assume that

$$2^k < k!$$

Note that since $k \geq 4$

$$2^{(k+1)} = 2^k \cdot 2 < (k!)2$$

$$< (k!)(k+1) = (k+1)!$$

↑

since $2 < k+1$.

Hence $P(k+1)$ is true. By

induction $P(n)$ is true for

all $n \geq 4$.

Sometimes the Induction

Principle can be expressed

as follows.

Let S be a subset of \mathbb{N}

such that

(1) $P(1)$ is true.

(2) For every $k \in \mathbb{N}$,

if $P(1), \dots, P(k)$ are all true, then $P(k+1)$ is true,

Then $P(n)$ is true for all $n \in \mathbb{N}$.

This is sometimes called the Principle of Strong Induction.

Ex. Suppose a sequence

$\{x_n\}$ is defined by

$$x_1 = 1, \quad x_2 = 2 \quad \text{and}$$

$$x_{n+2} = \frac{1}{2} (x_{n+1} + x_n).$$

Use Strong Induction to

show that

$$1 \leq x_n \leq 2, \quad \text{all } n \in \mathbb{N}.$$

Let $P(n)$ be the statement
that $1 \leq x_n \leq 2$.

Note that $P(1)$ and $P(2)$
both hold by hypothesis.

Now let $k \in \mathbb{N}$ with $k \geq 2$,
and suppose that $P(j)$ is
true for all $j \leq k$, i.e.,

$$1 \leq x_j \leq 2, \quad \text{if } 1 \leq j \leq k.$$

18

Then $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$$\leq \frac{1}{2}(2+2) = 2$$

by strong induction
hypothesis.

and

$$x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$$

$$\geq \frac{1}{2}(1+1) = 1$$

by strong ind.
hypothesis

Hence $1 \leq x_{k+1} \leq 2$,

which shows that $P(k+1)$

is true. Thus the Strong

Induction Principle

implies that $P(n)$ is

true for all $n \in \mathbb{N}$.

Conjecture a formula for

$$1 + 3 + \dots + (2n-1).$$

← Set $S_n =$

$$n=1, S_1 = 1$$

$$n=2, S_2 = 1 + 3 = 4$$

$$n=3, S_3 = 1 + 3 + 5 = 9$$

$$n=4, S_4 = 1 + 3 + 5 + 7 = 16$$

It seems that $S_n = n^2$.

Prove by induction.

When $n=1$, it is obvious
that $S_1 = 1 = 1^2$.

Now assume that $S_n = n^2$ (1)

$$\begin{aligned} S_{n+1} &= (1 + 3 + \dots + (2n-1)) + (2n+1) \\ &= n^2 + (2n+1) \\ &= (n+1)^2. \end{aligned}$$

This shows that (1) is true
for $n+1$.

By induction,

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

for all $n \in \mathbb{N}$.

Ex. Find all $n \in \mathbb{N}$ such that

$$n^2 < 2^n.$$

Try $n=1$	$1 < 2^1$	✓
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$n=2$	$4 < 4$	✗
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$n=3$	$9 < 8$	✗
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$$n=4 \quad 16 < 2^4 = 16 \quad \times$$

$$n=5 \quad 25 < 2^5 = 32 \quad \checkmark$$

We prove $n^2 < 2^n$ for $n \geq 5$.

We already proved this if $n=5$.

Assume $n^2 < 2^n$ for $n \geq 5$

$$(n+1)^2 = \frac{(n+1)^2}{n^2} \cdot n^2 = \left(\frac{n+1}{n}\right)^2 2^n$$

↑ inductive hypothesis.

If $n \geq 5$,

$$\left(1 + \frac{1}{n}\right) \leq 1 + \frac{1}{5} \leq \frac{6}{5}.$$

$$\therefore \left(\frac{n+1}{n}\right)^2 \leq \left(\frac{6}{5}\right)^2 \leq \frac{36}{25} < 2.$$

Hence

$$\left(\frac{n+1}{n}\right)^2 2^n < 2 \cdot 2^n = 2^{n+1}. \quad \checkmark$$

We conclude by induction

that $n^2 < 2^n$ if $n \geq 5$

$n = 1$ or if $n \geq 5$