Def'n. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$.

We say that f is uniformly continuous on A if for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $x_1, x_2 \in A$ are any numbers satisfying $|x_1 - x_2| < \delta(\varepsilon)$, then $|f(x_1) - f(x_2)| < \varepsilon$.

The point is that if we want to guarantee that $|f(x_1) - f(x_2)| < \varepsilon$, it suffices to choose δ sufficiently small, say $|x_1 - x_2| < \delta(\varepsilon)$.

Thm. If $I = [a, b]$ is a closed bounded interval, and f is continuous on I, then f is uniformly continuous on I.

Pf. If f is <u>not</u> uniformly continuous on I, then there is a number $\varepsilon_0 > 0$, such that for any number $\delta > 0$, there are numbers $u = u(\delta)$ and $v = v(\delta)$ such that $|u(\delta) - v(\delta)| < \delta$, but that $|f(u(\delta)) - f(v(\delta))| \geq \varepsilon_0$

In fact, for every $n \in N$, there are numbers u_n and v_n in I such that $|u_n - v_n| < \frac{1}{n}$, and that $|f(u_n) - f(v_n)| \geq \varepsilon_0$. (1)

Since I is bounded, the Bolzano-Weierstrass Thm implies that the sequence (u_n) has a subsequence

(u_{n_k}) that converges to a number x in $\mathbb{R}$.

Since $a \le u_{n_k} \le b$ for all $k = 1, 2, \ldots,$ it follows (from Thm 3.2.6) that

$x = \lim_{k \to \infty} u_{n_k}$ also is in $[a, b]$.

Note that

$$(v_{n_k} - x) = (v_{n_k} - u_{n_k}) + (u_{n_k} - x)$$

We know $|v_n - u_n| < \frac{1}{n} \to 0$

In particular, $\lim (v_{n_k} - u_{n_k}) = 0$

In addition, we know that $\lim (u_{n_k} - x) = 0.$ We conclude that $\lim v_{n_k} = x.$ Thus, it is clear that both u_{n_k} and v_{n_k} approach x. Since f is continuous at x, both $f(u_{n_k})$ and $f(v_{n_k})$ converge to $f(x)$, i.e.,

$$\left.\begin{aligned} &\lim \left(f(u_{n_k}) - f(x)\right) = 0 \\ &\text{and} \\ &\lim \left(f(v_{n_k}) - f(x)\right) = 0. \end{aligned}\right\} \quad (2)$$

Note that

$$|f(u_{n_k}) - f(v_{n_k})|$$

$$= \left|\left(f(u_{n_k}) - f(x)\right) - \left(f(v_{n_k}) - f(x)\right)\right|$$

$$\leq |f(u_{n_k}) - f(x)| + |f(v_{n_k}) - f(x)|$$

Combining this with (2), and

taking the limit as $k \to \infty$,

we conclude that

$$\lim |f(u_{n_k}) - f(v_{n_k})| = 0.$$

Replacing n by n_k in (1),

we get $|f(u_{n_k}) - f(v_{n_k})| \geq \varepsilon_0$,

which obviously is a

controdiction. Thus

f is uniformly continuous

on $I = [a, b]$.

Lipschitz Functions

Definition. Let $A \subseteq \mathbb{R}$ and let $f: A \longrightarrow \mathbb{R}$. If there is a constant $K > 0$, such that

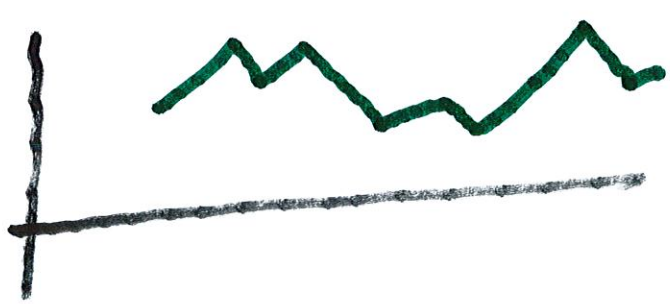
$$|f(x) - f(u)| \leq K|x - u|, \qquad (3)$$

for all $x, u \in A$, then

f is said to be a Lipschitz function on A.

Geometrically, the Lipschitz condition can be written as

$$\left| \frac{f(x)-f(y)}{x-y} \right| \le K$$



Thus, the slopes of all the segments joining two points on the graph of $y = f(x)$ are bounded by a constant K.

Thm. If $f: A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous

Pf If (3) is true, then given $\varepsilon > 0$, we can take $\delta = \varepsilon/K$. If $x, u \in A$ satisfy $|x-u| < \delta$, then

$$|f(x) - f(u)| \le K|x-u|$$

$$\le K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Ex. The function $g(x) = \sqrt{x}$

is continuous on $[0,1]$,

but it is not Lipschitz,

because if

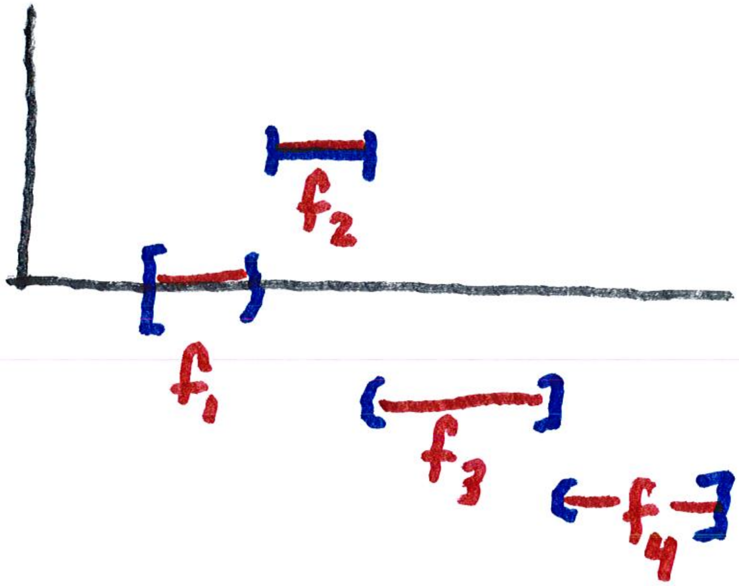
$$|g(x) - g(0)| \le |K(x-0)| = Kx,$$

then $\sqrt{x} \le Kx$ for all $x \in [0,1]$.

Thus $1 \le K\sqrt{x}$. But this

cannot happen if x is small in $[0,1]$

Def'n. Let $I \subseteq \mathbb{R}$ be an interval and let $s : I \to \mathbb{R}$. Then s is called a step function if it has only a finite number of values. Moreover, on each interval, the step function takes on only one value in the interior of each interval.



Thm. Let $I = [a, b]$ be a closed bounded interval, and let $f: I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a step function $s_\varepsilon : \bar{I} \to \mathbb{R}$

such that $|f(x) - s_\varepsilon(x)| < \varepsilon$

for all $x \in \bar{I}$.

Pf. The function f is

uniformly continuous, so

given $\varepsilon > 0$, there is a

number $\delta(\varepsilon)$ such that

if $x, y \in \bar{I}$ and $|x-y| \leq \delta$, then $|f(x) - f(y)| < \varepsilon$.

Let $I = [a, b]$ and let m be sufficiently large so that $h = (b-a)/m < \delta(\varepsilon)$

Now we divide $[a, b]$ into m disjoint intervals of length h.

$$a = x_0 < x_1 \ldots < x_{m-1} < x_m = b.$$

where $x_i - x_{i-1} = h = \frac{b-a}{m}$.

Now define

$s_\varepsilon(x) = f(a+kh)$, for all

$x \in I_k$, $k = 1, \ldots, m$,

so s_ε is constant on each

interval (The value of s_ε

on I_k is the value of f

at the right endpoint of I_k).

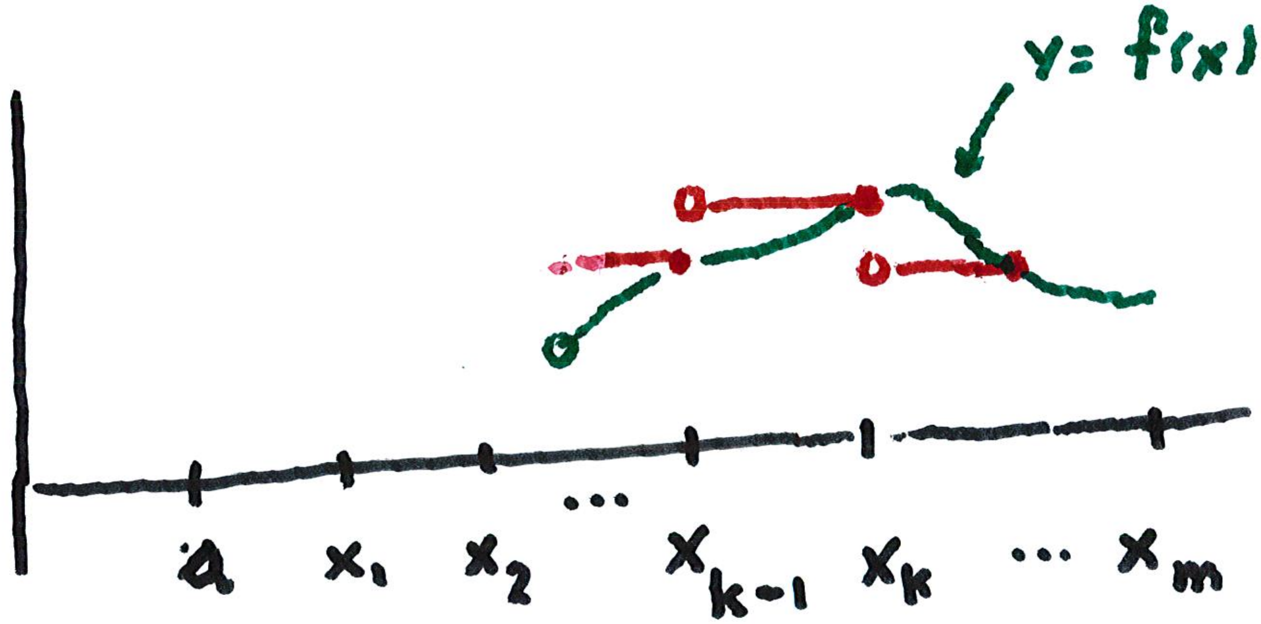
Hence, if $x \in I_k$, then

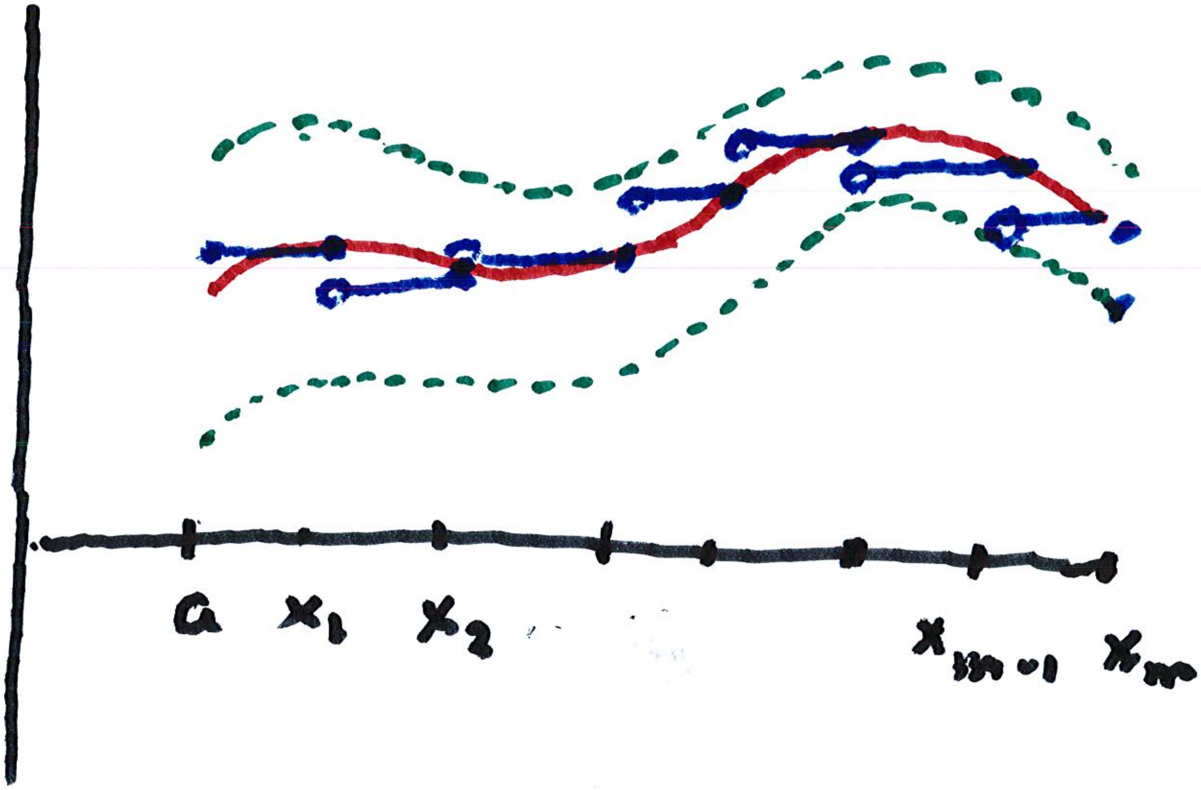
$$|f(x) - s_\varepsilon(x)| = |f(x) - f(a+kh)|$$

$$< \varepsilon .$$

Hence $|f(x) - s_\varepsilon(x)| < \varepsilon$

for all $x \in I$.




a
x_1
x_2
x_{m-1}
x_m

5.4.2 Nonuniform Continuity Criterion.

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is not uniformly continuous

(ii) There is an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and

$|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Ex. Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Set $x_n = n + \frac{1}{n}$ and $u_n = n$

Then

$$|f(x_n) - f(u_n)| = \left(n + \frac{1}{n}\right)^2 - n^2$$

$$= n^2 + 2n \cdot \frac{1}{n} + \frac{1}{n^2} - n^2$$

$$= 2 + \frac{1}{n^2} > 1$$ If we set

$\varepsilon_0 = 1$, then f is NOT uniformly continuous.