6.4 Taylor Series.

Suppose we write a polynomial p

as $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

Set $x = 0 \longrightarrow \underline{p(0) = a_0}$

Diff:

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}$

Set $x = 0 \rightarrow \underline{p'(0) = a_1}$

Diff:

$p''(x) = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \dots$

Set $x = 0$ $\underline{p''(0) = 2 \cdot a_2}$

Keep going: we get

$$p^{(k)}(0) = k(k-1)\ldots 2\cdot 1\, a_k$$

Solve for a_k:

$$a_k = \frac{p^{(k)}(0)}{k!}$$

If we write

$$p(x) = a_0 + a_1(x-a) + a_2(x-a)^2$$

$$\ldots + a_n(x-a)^n,$$

then $a_k = \dfrac{p^{(k)}(a)}{k!}$

Now suppose that f is a function (not necessarily a polynomial) such that $f^{(1)}(a), \ldots, f^{(n)}(a)$ all exist.

We define $a_k = \dfrac{f^{(k)}(a)}{k!}$ and

$$P_{n,a}(x) = a_0 + a_1(x-a) + \ldots a_n(x-a)^n$$

$P_{n,a}$ is the n-th Taylor polynomial of degree n for f at a.

The Taylor polynomial has been defined so that

$$P_{n,a}^{(k)}(a) = f^{(k)}(a) \text{ for } 0 \le k \le n.$$

It's the only polynomial of degree $\le n$ with this property.

Ex. Let $f(x) = \sin x$

$\sin 0 = 0$

$\sin'(0) = \cos 0 = 1$

$\sin''(0) = 0$

$\sin^{(3)}(0) = -\cos 0$
$= -1$

$\sin^{(4)}(0) = \sin 0 = 0$

From this point on, the derivatives repeat in a cycle of 4.

→ $a_k = \dfrac{\sin^{(k)}(0)}{k!}$

→ The Taylor polynomial $P_{2n+1,0}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Ex. Consider $f(x) = e^x$.

Since $f^{(n)}(0) = 1$ for all n,

we obtain

$$P_{n,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Ex. For $f(x) = \log x$, use $a = 1$.

$f(1) = 0$

$f'(1) = \frac{1}{x}$ $\quad \log'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad \log''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad \log'''(1) = 2$

In general,

$$\log^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k},$$

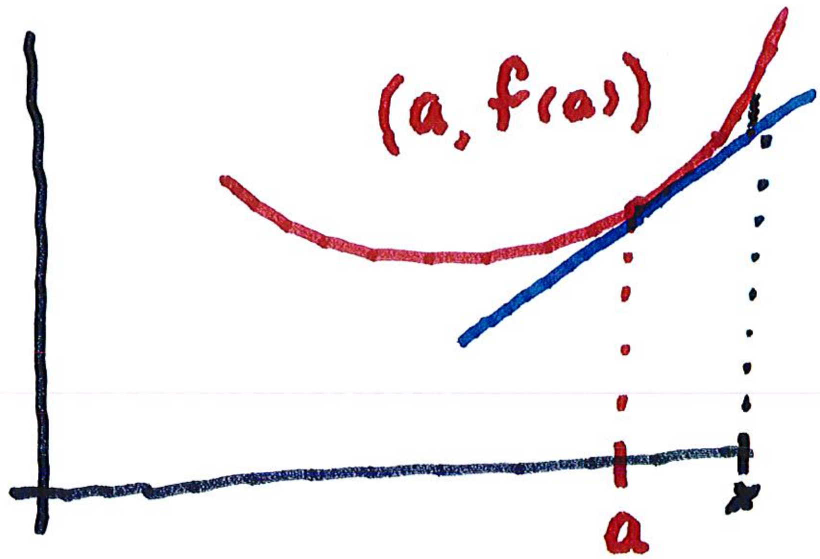
so $\log^{(k)}(1) = (-1)^{k-1}(k-1)!$

$\therefore P_{n,1}(x)$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots + \frac{(-1)^{n-1}(x-1)^n}{n}$$

Clearly $y = f(a) + f'(a)(x-a)$

is equal to the tangent line



The error $= |f(x) - P_{1,a}|$ is

smaller than $|x-a|$.

We will see that

$|f(x) - P_{n,a}(x)|$ is much smaller

than $|x-a|^n$

We want a formula for the error:

$$R_n(x) = f(x) - P_n(x)$$

Part 2 of the Fundamental Thm. of Calculus is

$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

We integrate by parts:

$u = f'(t) \quad dv = 1 \cdot dt$

$du = f''(t)\,dt \quad v = t - x$

$$= f'(t)(t-x)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt$$

$$= f'(x)\cdot 0 - f'(a)(a-x) + \int_a^x f''(t)(x-t)\,dt$$

Hence,

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt$$

By repeatedly increasing by parts,

we get

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + \dots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

$$+ R_n(x), \quad \text{where}$$

$$R_n(x) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

If we set $u = f^{(n+1)}(t)$ and

$dv = \frac{(x-t)^n}{n!} dt$, then we get

$du = f^{(n+2)}(t)\,dt$ and $v = -\frac{(x-t)^{n+1}}{(n+1)!}$

Hence, $R_n(x) = \int_a^x f^{(n+1)}(t) \frac{(x-t)^{n+1}}{n!} dt$

$$= f^{(n+1)}(t)(-1)\frac{(x-t)^{n+1}}{(n+1)!}\Bigg|_a^x$$

$$- \int_a^x f^{(n+2)}(t)(-1)\frac{(x-t)^{n+1}}{(n+1)!} dt$$

$$= -\int_a^x f^{(n+2)}(t)\,(-1)\,\frac{(x-t)^{n+1}}{(n+1)!}\,dt$$

$$= \underbrace{f^{(n+1)}(a)\,\frac{(x-a)^{n+1}}{(n+1)!} + \int_a^x f^{(n+2)}(t)\,\frac{(x-t)^{n+1}}{(n+1)!}\,dt}_{R_{n+1}(x)}$$

It follows by Induction that

$f(x) = P_{n,a}(x) + R_{n,a}(x)$ holds

for all $n \in N$.

The above formula for $R_n(x)$

is called the "integral form"

of the error.

In order to estimate it,

set $M_{n+1} = \sup_{t \in [a,x]} \{ |f^{(n+1)}(t)| \}$.

This gives

$$|R_n(x)| \leq \int_a^x |f^{(n+1)}(t)| \frac{(x-t)^n}{n!} dt$$

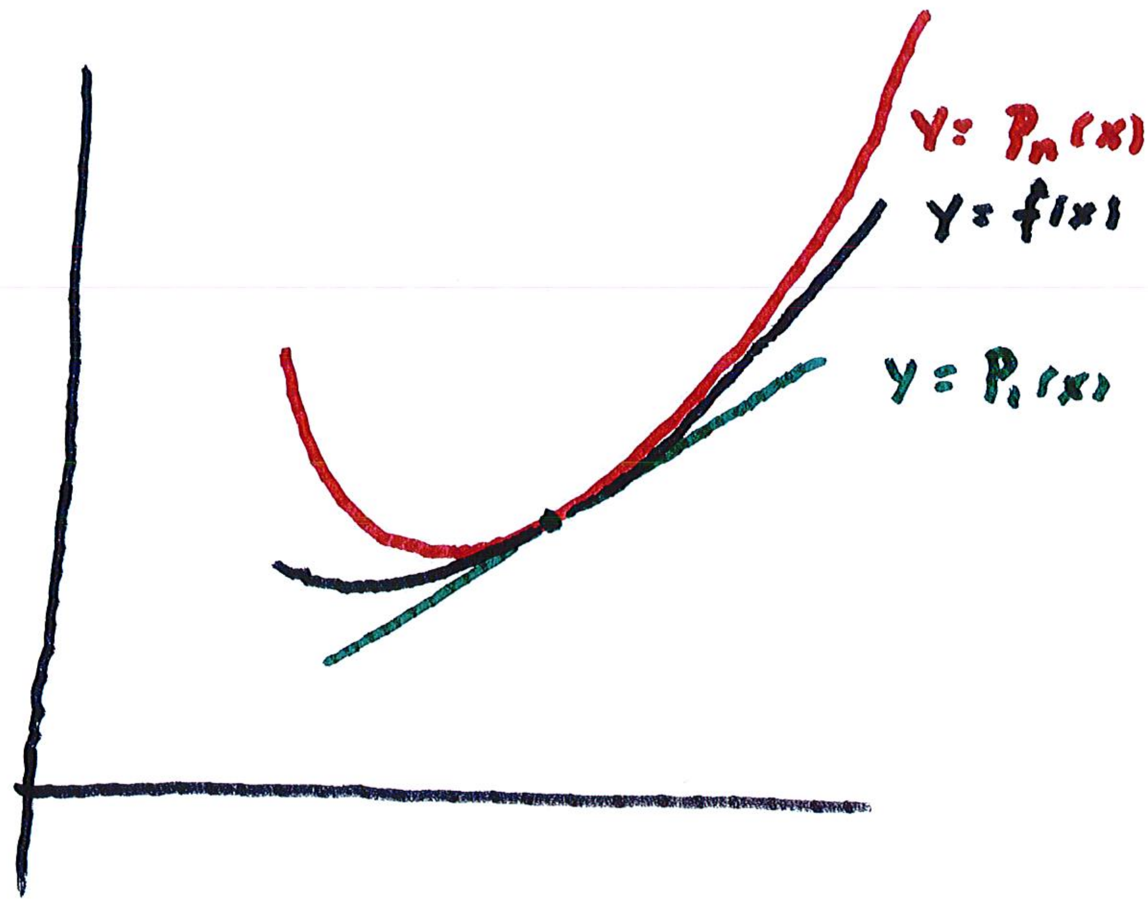
$$\leq M_{n+1} \int_a^x \frac{(x-t)^n}{n!}\,dt$$

$$= -M_{n+1} \left. \frac{(x-t)^{n+1}}{(n+1)!} \right|_{t=a}^{t=x}$$

$$= M_{n+1} \frac{|x-a|^{n+1}}{(n+1)!}$$

Thus, we've showed

$$|R_n(x)| \leq \frac{M_{n+1}\,|x-a|^{n+1}}{(n+1)!}$$



Note that $\cancel{|R_n}$ if x is close to a, then $|x-a|^{n+1}$ is much smaller than $|x-a|^n$.

Let $P_{2N+1}(x) = x - \frac{x^3}{3!} \cdots \pm \frac{x^{2N+1}}{(2N+1)!}$

converges to $\sin x$ as $N \to \infty$.

In fact, note that the

$|\sin^{(n)} x| \le 1$.

Hence $|R_{2N+1}(x)| \le \frac{1 \cdot |x|^{2N+2}}{(2N+2)!}$

$\to 0$ as $N \to \infty$ (for fixed $|x|$)

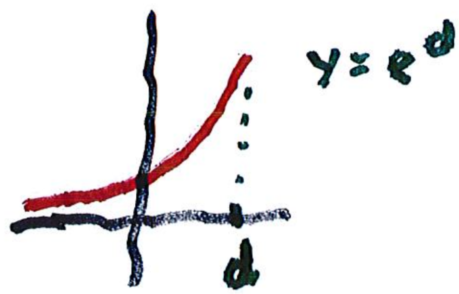
(Use the Ratio Test.)

$\therefore \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$

For $f(x) = e^x$, note that

$$\left| f^{(n+1)}(x) \right| \le e^x \le e^d \text{ if}$$

$|x| \le d$.



Hence $R_n(x) \le \dfrac{e^d \cdot d^{n+1}}{(n+1)!} \to 0$

by the Ratio Test.

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sometimes we can use substitution to find the power series of a function.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

(conv. if $|t| < 1$)

Let $t = -x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

OR, we can integrate or differentiate:

Ex. $\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$

$= \int_0^x (1 - t^2 + t^4 - t^6 + \cdots$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

[conv. if $|x|^2 < 1$

i.e., if $|x| < 1$.

L'Hopital's Rule again:

If $\lim_{x \to a^+} f(x) = 0$ and

$\lim_{x \to a^+} g(x) = 0$, then

we say $\lim \dfrac{f(x)}{g(x)}$ has the

indeterminate form $\frac{0}{0}$

Ex Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$ $\quad \frac{0}{0}$

$$\overset{\text{L'Hop.}}{=} \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

$\frac{0}{0}$

Find $\lim\limits_{x \to 0^+} \ln x \cdot x$

Must be a Quotient.

$$= \lim_{x \to 1} \frac{\ln x}{\frac{1}{x}} \overset{L'Hop}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$\overset{L'H}{=} \lim_{x \to 0} -x^2 = \underline{\underline{0}}$$

We can also look at indet. forms like $\lim\limits_{x \to \infty} \frac{x^2}{e^x} = \lim \frac{2x}{e^x}$ $\frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{2}{e^x} = 0.$$

Evaluate $\lim \left(1+\frac{1}{n}\right)^n$

$$\lim_{x\to\infty} \log\left(1+\frac{1}{x}\right)^x = \lim x \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim \frac{1}{\left(1+\frac{1}{x}\right)} = \frac{1}{1+0} = 1$$

$$\therefore e^{\log\left(1+\frac{1}{x}\right)^x} = \left(1+\frac{1}{x}\right)^x = e^1 = \underline{\underline{e}}$$