1.3 Infinite Sets.

A set S is denumerable

if there is a bijection

$f : N \longrightarrow S$

If we write $x_n = f(n)$,

for all $n = 1, 2, \ldots,$ then

$$S = \{ x_n : n = 1, 2, 3, \ldots \},$$

where $x_j \neq x_k$ if $j \neq k$.

Ex. Some examples.

The set $E = \{2n : n \in \mathbb{N}\}$

of even natural numbers

is denumerable.

So is $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$

So is $P = \{2, 3, 5, 7, 11, \dots\}$

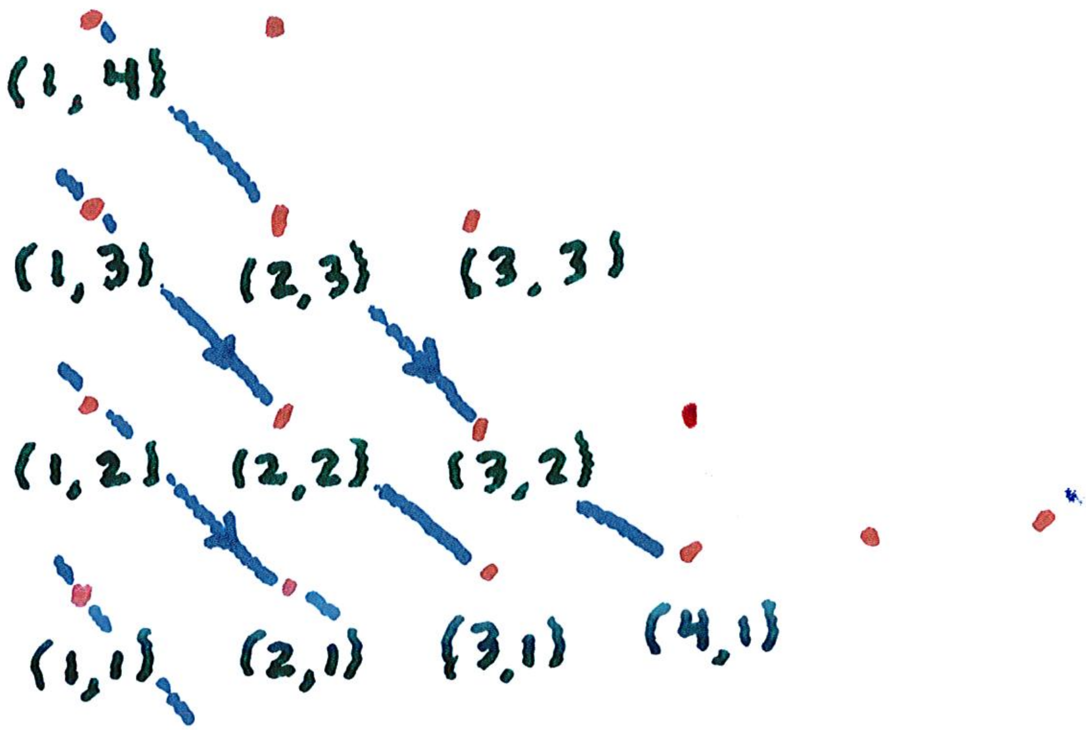
(the set of prime numbers).

$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

$$\begin{cases} f(n) = \dfrac{n}{2} & \text{if } n \text{ is even} \\ f(n) = -\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

is the formula for the bijection of $\mathbb{N}$ onto $\mathbb{Z}$.

Is $N \times N$ denumerable?



Follow first diagonal, then the second, then the third, etc. .

11				
7	12			
4	8	13		
2	5	9	14	
1	3	6	10	15

Using this method, let

$f(m,n)$ = value assigned

to (m,n).

Thus $f(1,1) = 1$ $\quad f(1,2) = 2$

$f(2,1) = 3.$ $\quad f(1,3) = 4$

$\ldots f(4,1) = 10, \quad \ldots$

Sum of first 2 diagonals

$= 1 + 2 = 3$ $\quad f(2,1) = 3$

Sum of k diagonals is

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

$$f(k,1) = \frac{k(k+1)}{2}.$$

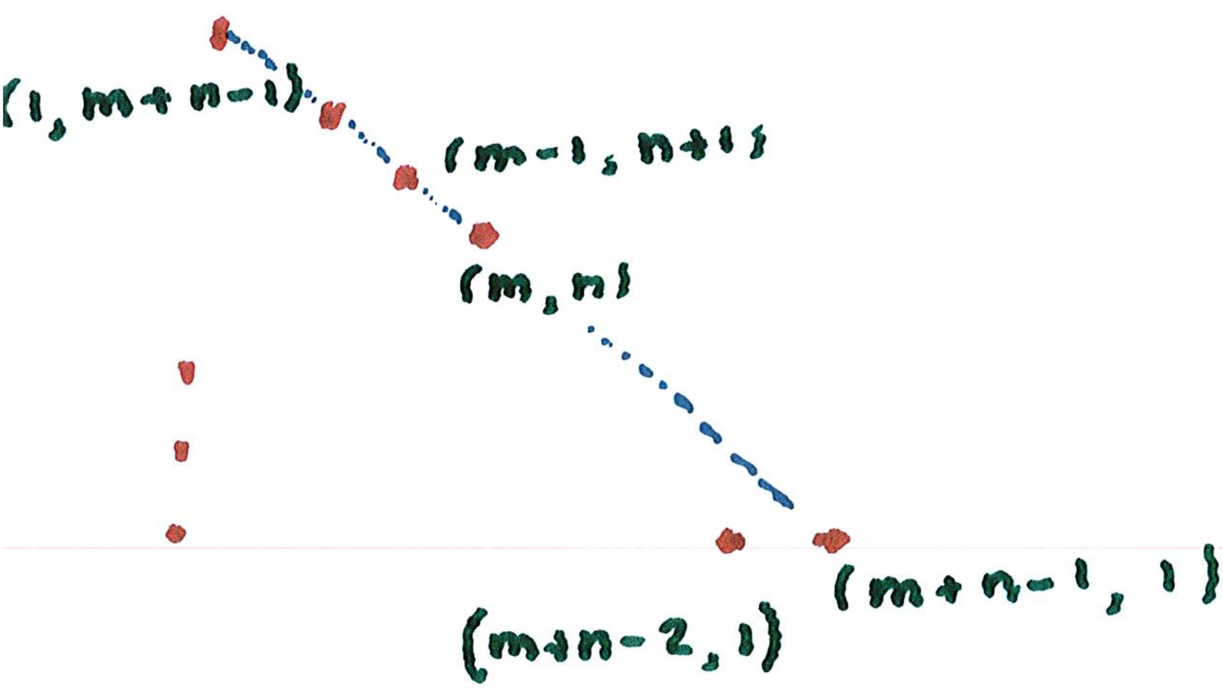
We see that the endpoints of

$(m+n-1)$-th diagonal are

$(1, m+n-1)$ and $(m+n-1, 1)$.

Hence the predecessor of

$(1, m+n-1)$ is $m+n-2$.



Hence,

$$
\begin{aligned}
f(m,n) &= f(m-1,\ n+1) + 1 \\
&= f(m-2,\ n+2) + 2 \\
&\ \vdots \\
&= f(1,\ m+n-1) + (m-1) \\
&= f(m+n-2,\ 1) + m
\end{aligned}
$$

$$
f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m
$$

Observe that as we move along the path, $f(m,n)$ increases by 1 with each step. Therefore,

$f: N \times N \to N$ is 1-to-1 and onto

It follows that f has an inverse $g: N \to N \times N$ that is also 1-to-1 and onto.

g satisfies

$g(1) = (1,1)$

$g(2) = (1,2)$

$g(3) = (2,1)$

$g(4) = (1,3)$, etc.

In general

$$g(k) = \big(m(k),\ n(k)\big)$$

for $k = 1, 2, \ldots$

Now define a

function $\pi(m,n) = \frac{m}{n}$

and also define

$$h(k) = \pi(g(k)) = \frac{m(k)}{n(k)}$$

This is the k-th positive rational number at the k-th point on the path.

Thus we obtain a function $h: N \to Q^+$

that is onto but

not 1-to-1.

We want to modify h

to make it 1-to-1 and onto.

Idea: We have a path

$h: \mathbb{N} \longrightarrow \mathbb{Q}^+$ that runs

through all rational numbers

We should delete all

rational numbers that

already occur on the

list.

$\frac{1}{5}$ 10 $\quad \frac{2}{5} \quad \frac{3}{5}$

$\frac{1}{4}$ 6 $\quad \frac{2}{4}$ x $\quad \frac{3}{4} \quad \frac{4}{4}$

$\frac{1}{3}$ 4 $\quad \frac{2}{3}$ 7 $\quad \frac{3}{3}$ x $\quad \frac{4}{3}$

$\frac{1}{2}$ 2 $\quad \frac{2}{2}$ x $\quad \frac{3}{2}$ 8 $\quad \frac{4}{2}$ x

$\frac{1}{1}$ 1 $\quad \frac{2}{1}$ 3 $\quad \frac{3}{1}$ 5 $\quad \frac{4}{1}$ 9 $\quad \frac{5}{1}$ 11

We delete $\frac{m}{n}$ if m and n have a common factor $q > 1$, i.e., if the rational number $\frac{m}{n}$ already occurs on the list

Thus, we obtain a function

$H : N \longrightarrow Q^+$ that is 1-to-1

and onto:

$H(1) = \frac{1}{1}$ $\qquad$ $H(7) = \frac{2}{3}$

$H(2) = \frac{1}{2}$ $\qquad$ $H(8) = \frac{3}{2}$

$H(3) = \frac{2}{1}$ $\qquad$ $H(9) = \frac{4}{1}$

$H(4) = \frac{1}{3}$ $\qquad$ $H(10) = \frac{1}{5}$

$H(5) = \frac{3}{1}$ $\qquad$ $H(11) = \frac{5}{1}$

$H(6) = \frac{1}{4}$ $\qquad$ $H(12) = \frac{1}{6}$, etc.

Thus, the function

$H: N \longrightarrow Q^+$ provides a list

of all positive rational

numbers such that each

rational number exactly

once on the list. Thus,

H is 1-to-1 and onto.

Hence Q^+ is denumerable.

If we write $\mathbb{H}(k) = r_k$.

for $k = 1, 2, \ldots,$ then

$$Q^+ = \{ r_1, r_2, r_3, \ldots \}$$

Now we write

$$\begin{array}{lll}
(r_1, r_4) & & \\
(r_1, r_3) & (r_2, r_3) & (r_3, r_3) \\
(r_1, r_2) & (r_2, r_2) & (r_3, r_2) \\
(r_1, r_1) & (r_2, r_1) & (r_3, r_1)
\end{array}$$

This is a list Q_2^+ of all ordered pairs of positive rational numbers. We conclude Q_2^+ is also deumerable. Letting R_k be the k-th element of this list, consider

(R_1, r_4)		
(R_1, r_3)	(R_2, r_3)	(R_3, r_3)
(R_1, r_2)	(R_2, r_2)	(R_3, r_2)
(R_1, r_1)	(R_2, r_1)	(R_3, r_1)

This provides a list of all ordered triples of positive rationals.

Hence

$\mathbb{Q}_3^+$ is denumerable.

Sets can be arbitrarily large: For any set S, let

$\wp(S)$ be the set of all subsets of S.

Cantor's Thm:
There does NOT exist a map $\varphi : S \longrightarrow \wp(S)$ that is onto.

Proof. Suppose

$$\varphi : S \longrightarrow \mathcal{P}(S)$$

is a surjection.

Since $\varphi(x)$ is a subset of S. Either x belongs to $\varphi(x)$ or it does not belong to $\varphi(x)$. We let

$$D = \left\{ x \in S : x \notin \varphi(x) \right\}$$

Since φ is a surjection,

there exists $x_0 \in S$
such that $\varphi(x_0) = D$.

There are 2 cases:

1. Suppose $x_0 \in D$.

   Then $x_0 \in \varphi(x_0)$.

   By definition of D,

   $x_0 \notin D$. Contradiction

2. Suppose $x_0 \notin D$.

Then $x_0 \notin \varphi(x_0)$.

By definition of D,

$x_0 \in D$. Contradiction.

Ex. Suppose $S = \{a, b, c\}$

$$\wp(S) = \Big\{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\} \text{ and } \{a,b,c\} \Big\}$$

S has 3 elements,

$\mathcal{P}(S)$ has 8 elements.

There does not exist

a surjection from

S onto $\mathcal{P}(S)$.