Fundamental Theorem of Calculus, Part 1.

Let f be a continuous function on a closed bounded interval J.

Given a number $a \in J$, we define a function F on J as follows: $F(x) = \int_a^x f$, all $x \in J$.

Then F is continuous on J, and at each $x_0 \in J$, F is differentiable and $F'(x_0) = f(x_0)$.

Proof. Since f is continuous on J, it follows that f is bounded, i.e. $|f(x)| \leq M$, if $x \in J$.

Hence, if x and y are two points with, say $x \leq y$, then

$$F(y) - F(x) = \int_a^y f - \int_a^x f = \int_x^y f,$$

$$|F(y) - F(x)| = \left| \int_x^y f \right| \leq \int_x^y |f|$$

$$\leq \int_x^y M = M(y-x)$$

Thus, f is Lipschitz on J

which implies that F is

uniformly continuous on J.

Now suppose that f is

right-continuous at x_0, where

$x_0 \in J$. Consider $x \in J$ with

$x > x_0$. Then

$$F(x) - F(x_0) = \int_{x_0}^{x} f(t)\,dt$$

and

$$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^{x} f(x_0)\,dt.$$

From these two equations we get

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)$$

$$= \frac{1}{x - x_0} \int_{x_0}^{x} [f(t) - f(x_0)]\,dt$$

and thus,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$\leq \frac{1}{x - x_0} \int_{x_0}^{x} |f(t) - f(x_0)| \, dt .$$

Let $\varepsilon > 0$ be given. Since f is right-continuous at x_0, there exists a $\delta > 0$ so that for all $t \in J$,

$$x_0 < t < x_0 + \delta \Rightarrow |f(t) - f(x_0)| \leq \varepsilon$$

Thus, if $x_0 < x < x_0 + \delta$, then

$$\int_{x_0}^{x} | f(t) - f(x_0) | \, dt$$

$$\leq \int_{x_0}^{x} \varepsilon \, dt = \varepsilon (x - x_0),$$

so that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \varepsilon.$$

This proves that

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Similarly, if f is left-continuous at x_0, then it can be shown that

$$F'(x_0^-) = f(x_0).$$

It follows that if f is continuous at x_0 in the usual

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two-sided sense and

$F'(x_0) = f(x_0)$.

Corollary. If f is continuous on J, then f has an antiderivative F on J.

To say that F is an antiderivative means $\underline{F'(x) = f(x)}$, for all $x \in J$

This corollary makes it much easier to compute indefinite integrals:

Suppose we want to compute $\int_1^2 t^2\,dt$. Let $f(x) = x^2$ and set $F(x) = \int_1^x t^2\,dt$

Then FTC, part 1 states that $F'(x) = x^2$.

Note that $\frac{x^3}{3}$ also

satisfies $\left(\frac{x^3}{3}\right)' = x^2$

One of the corollaries of

the Mean Value Theorem

states that if two functions

$F(x)$ and $G(x)$ satisfie

$F'(x) = G'(x)$, then F and

G differ by a constant C

For us, this means that

$$F(x) = \frac{x^3}{3} + C.$$

If we set $x=1$, then

$$0 = F(1) = \frac{1^3}{3} + C,$$ so

$C = -\frac{1}{3}$. We conclude that

$$F(x) = \frac{x^3}{3} - \frac{1}{3}.$$

If $x = 2$, then

$$F(2) = \frac{2^3}{3} - \frac{1}{3}$$

More generally, suppose we want to compute $\int_a^b f(t)\,dt$.

The Fund Thm of Calculus states that

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

Suppose $F(x)$ satisfies

$$F'(x) = f(x).$$

Then both $\int_a^x f(t)\,dt$ and $F(x)$

have the same derivative.

Hence there is a constant c

so that

$\int_a^x f(t)\,dt = F(x) + c$. Since

with $x = a$, $0 = \int_a^a f(t)\,dt = F(a) + c$

$\rightsquigarrow c = -F(a)$, we get (if $x = b$)

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Thus, the Fund. Thm. of Calculus (Part 1) states that if f is continuous at x, then

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

(Part 2) states that if $f(t)$ is continuous at all t with $a \le t \le x$, ~~and if $F(x)$ satisfies~~

F'~~$(x) = f(x)$, for all x with~~

~~$a \le x \le b$~~, then

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

for all x with $a \le x \le b$