We defined the Darboux integrable.

For a given partition P, we set

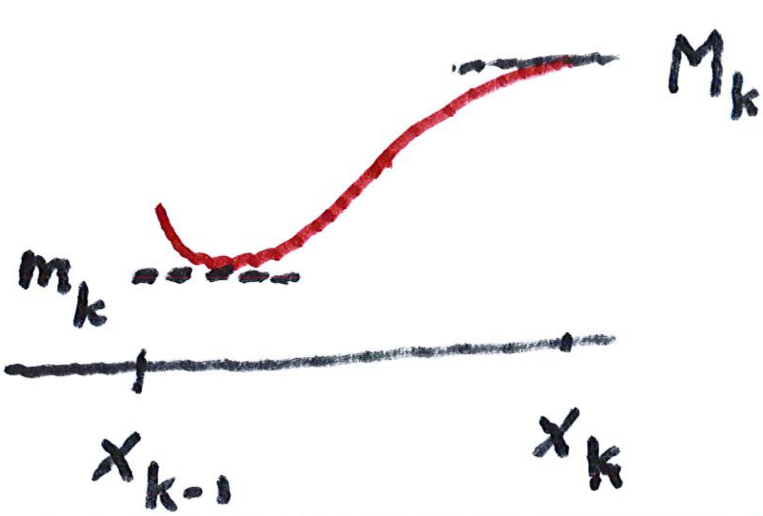
$$U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}),$$

$$L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}),$$

where

$$M_k = \sup\{f(x) : x \in I_k\}$$

$$m_k = \inf\{f(x) : x \in I_k\}$$



We also defined

$$U(f) = \inf\left\{ U(f, P) : \text{for all partitions } P \text{ of } [a, b] \right\}$$

and

$$L(f) = \sup\left\{ L(f, P) : \text{for all partitions } P \text{ of } [a, b]. \right\}$$

Finally we define f to be (Darboux) integrable on $[a,b]$ if $L(f) = U(f)$, and we define $\int_a^b f$ or $\int f = L(f)$.

We want to simplify the task of determing when f is integrable. For this, we have

Integrability Criterion

Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be bounded. Then f is integrable if and only if for each $\varepsilon > 0$, there is a partition P_ε of I such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon. \tag{1}$$

Pf. We first assume f is integrable. We must find P_ε so that (1) holds

Since f is integrable, $L(f) = U(f)$. If $\varepsilon > 0$, then there is a partition P_1 so that $L(f, P_1) > L(f) - \frac{\varepsilon}{2}$ (2)

Similarly, there is a partition

P_2 so that

$$U(f, P_2) < U(f) + \frac{\varepsilon}{2}. \qquad (3)$$

If we let $P_\varepsilon = P_1 \cup P_2$, then

P_ε is a refinement of P_1

and P_2. Hence,

$$L(f) - \frac{\varepsilon}{2} < L(f : P_1) \leq L(f : P_\varepsilon)$$

$$\leq U(f : P_\varepsilon) \leq U(f : P_2) < U(f) + \frac{\varepsilon}{2}.$$

The first inequality becomes

$$-L(f: P_\varepsilon) < -L(f) + \frac{\varepsilon}{2} .$$

and the second becomes

$$U(f: P_\varepsilon) < U(f) + \frac{\varepsilon}{2} .$$

If we add these and use $U(f) = L(f)$, we obtain (1).

Now we assume there is a P_ε so that (1) holds. We must show that f is integrable.

For any partition P, we have

$L(f:P) \leq L(f)$, and

$U(f:P) \geq U(f)$. We can

write these as

$-L(f) \leq -L(f:P)$ and

$U(f) \leq U(f:P)$. Adding these:

$U(f) - L(f) \leq U(f:P) - L(f,P)$.

If we set $P = P_\varepsilon$, then (1) becomes

$$U(f) - L(f) < \varepsilon .$$

Since this is true for all ε, we conclude that

$$U(f) - L(f) \le 0.$$

Since we always have

$U(f) \ge L(f)$, or $U(f) - L(f) \ge 0$,

this shows $U(f) - L(f) = 0$

i.e. $U(f) = L(f)$, which implies that $U(f) = L(f)$, which means f is integrable, which proves the Integrability Criterion. We show how to use the Criterion:

Thm. If f is continuous on I, then f is integrable.

Pf. Since f is continuous on a closed bounded interval, f is uniformly continuous. For any $\varepsilon > 0$, there is a number $\delta > 0$ so that if x' and x'' are in I and $|x' - x''| < \delta$ then

$$|f(x') - f(x'')| < \frac{\varepsilon}{2(b-a)}.$$

Choose an integer $n > 0$,

so that $\frac{b-a}{n} < \delta$. Define

a partition P by

$a = x_0 < x_1 < \dots x_k < \dots x_n = b$.

where $x_k - x_{k-1} = \frac{b-a}{n} < \delta$.

Note that if $x \in [x_{k-1}, x_k]$,

then $|x - x_k| \le \frac{b-a}{n} < \delta$,

Hence $|f(x) - f(x_k)| < \frac{\varepsilon}{2(b-a)}$.

which means

$$f(x_k) - \frac{\varepsilon}{2(b-a)} < f(x) < f(x_k) + \frac{\varepsilon}{2(b-a)}$$

It follows that

$$M_k \le f(x_k) + \frac{\varepsilon}{2(b-a)} \qquad \text{and}$$

$$m_k \ge f(x_k) - \frac{\varepsilon}{2(b-a)},$$

which yields

$$M_k - m_k \le \frac{\varepsilon}{b-a}.$$

This implies that

$U(f : P) - L(f, P)$

$$= \sum_{k=1}^{n} M_k (x_k - x_{k-1}) - \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$$

$$\leq \sum_{k=1}^{n} \frac{\varepsilon}{(b-a)} (x_k - x_{k-1})$$

$$= \frac{\varepsilon}{(b-a)} (b-a) = \varepsilon$$

The Criterion implies that

f is integrable.

We can also allow f to have a finite number of discontinuities.

Theorem. Let $f: I \to \mathbb{R}$ be a bounded function. Let $E = \{c_0, \ldots, c_N\}$ be a distinct set of points in E with $c_0 < c_1 \ldots < c_N$, and assume that f is continuous at all points x in $[a,b]$, except for $x \in E$. Then f is integrable on $[a,b]$.

Pf. We can assume that a and b are in E. Thus $a = c_0$ and $b = c_N$. Let σ be a positive number such that

$$\sigma < \min\left\{ \frac{c_k - c_{k-1}}{2}, \; k = 1, 2, \ldots, N \right\}$$

and also that

$$\sigma < \frac{\varepsilon}{8M(N+1)},$$

where $|f(x)| \le M$ for all $x \in [a, b]$. Note that the first condition on σ implies that the intervals

$$[c_k - \sigma, c_k + \sigma], \text{ for } k = 0, 1, \ldots N$$

are all disjoint.

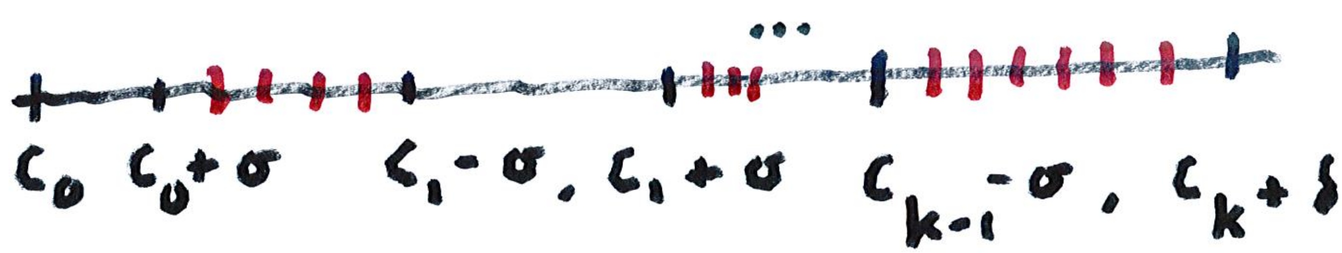
Since f is continuous on each interval, f is Darboux-integrable on each interval I_k; we can choose P_k so that

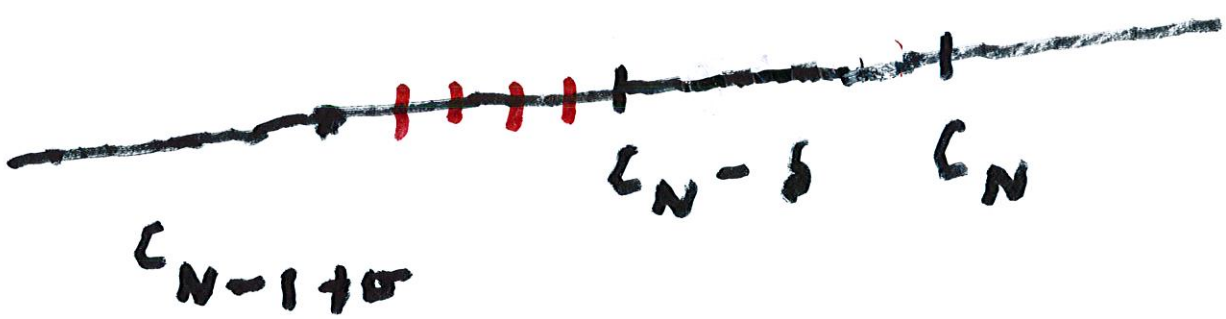
$$U(f, P_k) - L(f, P_k) < \frac{\varepsilon}{2N}$$

Now we form a partition

$$P = \bigcup_{k=1}^{N} P_k \cup \{a, b\}$$

Thus





If we let

$$J_0 = [c_0, c_0 + \sigma],$$

$$J_k = [c_k - \sigma, c_k + \sigma], \quad 1 \le k \le N-1$$

$$J_N = [c_N - \sigma, c_N], \quad \text{and}$$

and $M_k = \sup \{ f(x) : x \in J_k \}$

and $m_k = \inf \{ f(x) : x \in J_k \}$

for $k = 0, 1, \dots, N$

then $M_k \le M$

and $m_k \ge -M$, for

$k = 0, 1, \ldots, N$.

Hence $M_k - m_k \le 2M$.

Therefore,

$$U(f:P) - L(f:P)$$

$$\leq 2M\sigma + U(f:P_1) - L(f:P_1)$$

$$+ 2M\cdot 2\sigma + U(f:P_2) - L(f:P_2)$$

$$\ldots + 2M\cdot 2\sigma + U(f:P_N) - L(f:P_N)$$

$$+ 2M\cdot\sigma$$

$$\frac{M(N+1)4\varepsilon}{8M(N+1)} + \frac{N\varepsilon}{2N}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the Integrability Criterion implies that f is integrable.