Solutions of Differential Equations.

Let $F(x,y)$ be a continuous function on a domain $\underbrace{(a,b)\times(c,d)}$

We will assume that

$|F(x,y)| \le M$, for all $(x,y) \in R$

and also that F satisfies the Lipschitz condition

$$|F(x,s) - F(x,t)| \le C|s-t|.$$

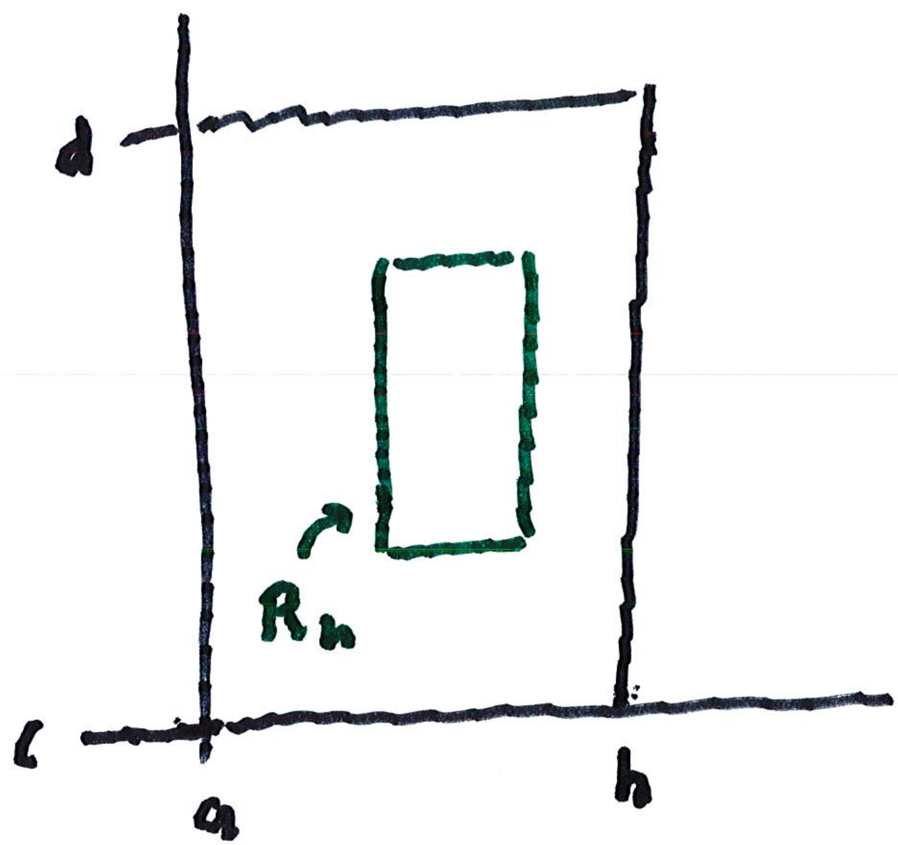
Let $x_0 \in (a,b)$ and $y_0 \in (c,d)$.

Choose $h > 0$ so that the

rectangle R_h

$$R_h = [x_0 - h, x_0 + h] \times [y_0 - Mh, y_0 + Mh]$$

is contained in R.



and we also need that

$Ch < 1$. If these equations

are not satisfied, we just

shrink h a bit

Under these conditions,

we will show that there is

a continuous function $y(x)$

for all $x \in [x_0 - h, x_0 + h]$,

such that

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt.$$

By the Fundamental Theorem

of Calculus, the righthand

is differentiable in x, so that it satisfies

$$y'(x) = F(x, y(x)).$$

Since $F(x, y(x))$ is continuous it follows that $y(x)$ is a continuously differentiable function that is a solution of the differential equation that satisfies $y(x_0) = y_0$.

Proof. We set $y_0(x) = y$ for all x in $[x_0 - h, x_0 + h]$.

Then we set

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0)\, dt$$

Note that

$$|y_1(x) - y_0| \le \left| \int_{x_0}^{x} F(t, y_0)\, dt \right|$$

$$= \int_{x_0}^{x} |F(t, y_0)|\, dt \le M|x - x_0|$$

$$\le Mh.$$

Hence $(x, y_1(x)) \in R_h$.

which shows that

$y_1(x)$ is well defined.

Now we define

$$y_2(x) = y_0 + \int_{x_0}^{x} F(t, y_1(t))\, dt,$$

which is again well defined.

Continuing, we define

for every $j = 1, 2, \ldots$

$$y_{j+1}(x) = y_0 + \int_{x_0}^{x} F(t, y_j(t))\, dt.$$

We will show that

$y_j(x)$ converges to a

function $y(x)$ as $j \to \infty$.

This will mean

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))\, dt,$$

which means $y(x)$ is a

solution of the equation.

Set $y_0(x) = y_0$. Then

This gives

$$y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0)\, dt$$

This gives

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} F(t, y_0)\, dt \right|$$

$$\leq \int_{x_0}^{x} |F(t, y_0)|\, dt$$

$$\leq Mh.$$

Next:

$$Y_2(x) = y_0 + \int_{x_0}^{x} F(t, y_1(t))\,dt,$$

which gives

$$| Y_2(x) - Y_1(x)$$

$$= \left| \int_{x_0}^{x} F(t, y_1(t))\,dt - \int_{x_0}^{x} F(t, y_0(t))\,dt \right|$$

$$\leq \int_{x_0}^{x} | F(t, y_1(t)) - F(t, y_0(t)) |\,dt$$

$$\leq \int_{x_0}^{x} C \cdot | y_1(t) - y_0(t) |\,dt$$

$$\leq C \cdot (Mh) \cdot h$$

$$= MCh^2 = M \cdot h \cdot Ch.$$

One can continue:

$$|y_3(x) - y_2(x)|$$

$$\leq MC^2 h^3 = M \cdot h \cdot (Ch)$$

and more generally.

$$|y_{j+1}(x) - y_j(x)| \leq MC^j h^{j+1}$$

$$\leq M \cdot h \cdot (Ch)^j.$$

We want to use the Cauchy Criterion. Thus, if

$0 \leq K < L$ are integers:

$$|Y_K(x) - Y_L(x)| \leq |Y_K(x) - Y_{K+1}(x)|$$

$$+ |Y_{K+1}(x) - Y_{K+2}(x)|$$

$$+ \cdots + |Y_{L-1}(x) - Y_L(x)|$$

$$\leq Mh \cdot \left([Ch]^K + [Ch]^{K+1} + \cdots + [Ch]^{L-1} \right).$$

Since $|Ch| < 1$, the geometric series $\sum_j |Ch|^j$ converges.

Hence, the right hand side of the inequality above is as small as we please, for K and L large, by the Cauchy Criterion for convergent series. If we let $L \to \infty$. then for all $x \in [x_0, x_0+h]$

we have

$$|y_K(x) - y(x)|$$

$$\leq Mh \cdot \left(|Ch|^K + |Ch|^{K+1} + \cdots \right)$$

$$= \frac{Mh\,|Ch|^K}{1 - Ch}.$$

Thus, it follows that

$y_K(x)$ converges <u>uniformly</u>

to $y(x)$.

Definition. A sequence of real-valued functions $f_1, f_2, \ldots$ is said to converge uniformly on a set S if and only if for all $\varepsilon > 0$ there is an integer $N > 0$, so that if $n > N$, then

$$|f(x) - f_n(x)| < \varepsilon.$$

Theorem. Suppose that $f_n \to f$ uniformly in some neighborhood of $x_0 \in \mathbb{R}$ and suppose that each function f_n is continuous on $[a, b]$. Then f is continuous on $[a, b]$.