Definition. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. If the sequence $(|a_n|^{\frac{1}{n}})$ is bounded, we set

$$\rho = \limsup \{ |a_n|^{1/n} \}.$$

If this sequence is not bounded we set $\rho = +\infty$. We define the radius of convergence of $\sum_{n=0}^{\infty} (a_n x^n)$ to be given by

$$R = 0, \quad \text{if } \rho = +\infty$$

$$= \frac{1}{\rho}, \quad \text{if } 0 < \rho < +\infty$$

$$= +\infty, \quad \text{if } \rho = 0.$$

At this point, we wish to recall what is meant by

$\limsup (c_n)$, where

(c_n) is a sequence of numbers.

If (b_n) is a bounded sequence of non-negative real numbers, then we set

$$B_1 = \sup \{ b_1, b_2, \dots \}$$

$$B_2 = \sup \{ b_2, b_3, \dots \}$$

$$B_3 = \sup \{ b_3, b_4, \dots \}$$

etc.

Clearly (B_n) is a decreasing sequence, because B_{n+1} is the supremum that is computed using a smaller set than the set used to compute B_n.

We now justify the term "radius of convergence".

Theorem. (Cauchy - Hadamard)

If R is the radius of convergence of the power series $\sum (a_n x^n)$, then the series is absolutely convergen- if $|x| < R$ and divergent if $|x| > R$.

Proof. We shall first treat the case where $0 < R < +\infty$.

If $0 < |x| < R$, then there exists a positive number $c < 1$ such that $|x| < cR$.

Therefore $\rho < c/|x|$

(recall that $\rho = \frac{1}{R}$)

and so it follows that if n is sufficiently large, then

$$|a_n|^{1/n} \leq \frac{c}{|x|}.$$

This is equivalent to the statement that

$$|a_n x^n| \le c^n$$

for all sufficiently large n.

Since $c < 1$, the absolute convergence of $\sum |a_n x^n|$ follows from the Comparison Test

If $|x| > R = 1/\rho$, then

there are infinitely many $n \in \mathbb{N}$ for which

$$|a_n|^{1/n} > \frac{1}{|x|}.$$

Therefore, $|a_n x^n| > 1$ for infinitely many n, so that the sequence $(a_n x^n)$ does not converge to zero.

Thm. Let R be the radius of convergence of $\sum (a_n x^n)$ and let K be a closed subset of the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.

Theorem. The limit of a power series is continuous on the interval of convergence.

A power series can be integrated term-by-term over any closed bounded interval contained in the interval of convergence