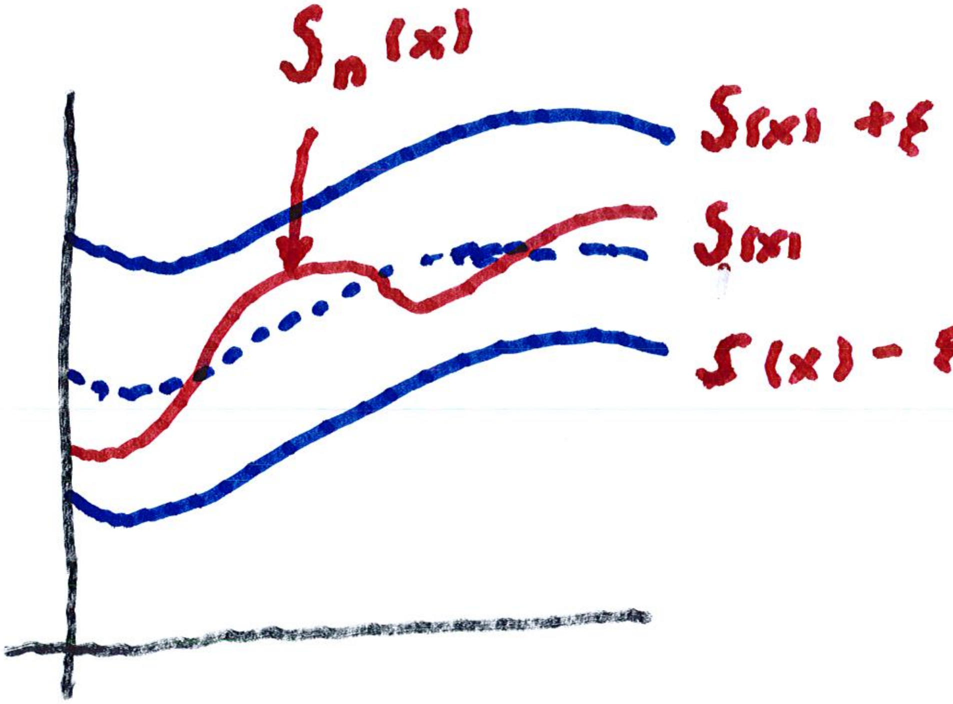
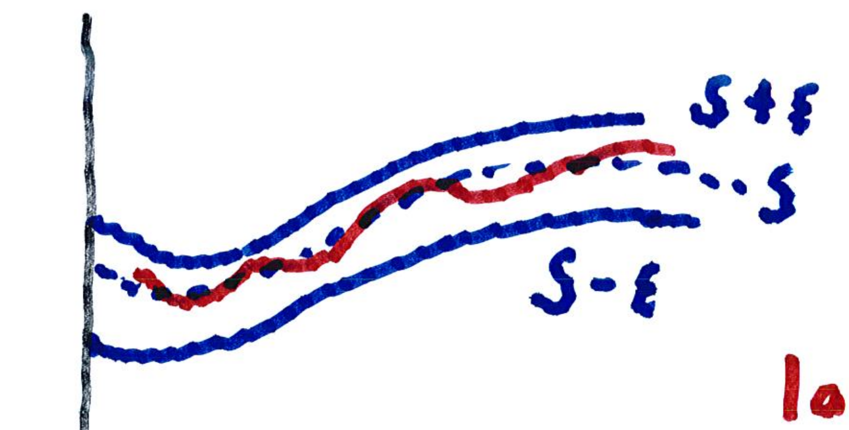
Def'n. Let $\{S_n\}$ be a sequence of functions on an interval I. We say $\{S_n\}$ converges uniformly to a function S in I if for every $\varepsilon > 0$ there is an $N(\varepsilon, I)$ where N is independent of the particular x in I, so that $|S_n(x) - S(x)| < \varepsilon$ if $n > N$ for all x in I.



$S_n \to S$ uniformly



n is much larger,

$S_n(x)$ closer to $S(x)$

There is also a Cauchy criterion for uniform convergence:

Let $\{S_n\}$ be a sequence of functions defined on an interval I. In order that the sequence converge uniformly in I, it is sufficient that for each $\varepsilon > 0$ there be an $N(\varepsilon)$ (indepent of x in I for which

$$|S_n(x) - S_m(x)| < \varepsilon \quad \text{if } n > N \text{ and } m > N.$$

Here is a test for uniform convergence of series.

Weierstrass M-Test.

Suppose $\{u_n\}$ is a sequence of functions defined on an interval I, and there is a sequence of positive constants

M_n with $|u_n(x)| \leq M_n$

for all x in I and all n.

If the series $\sum_{n=1}^{\infty} M_n < \infty$, i.e.,

converges, then the series

$\sum_{n=1}^{\infty} u_n$ converges uniformly

in I.

Proof. Since $\sum_{n=1}^{\infty} M_n$ converges,

for any $\varepsilon > 0$ there is an

$N(\varepsilon)$ for which $\displaystyle\sum_{k=n+1}^{m} M_k < \varepsilon$ if $n > N$.

Then if $\displaystyle S_n(x) = \sum_{k=n+1}^{m} u_k(x)$,

we have for all x in I,

$$|S_m(x) - S_n(x)| = \left|\sum_{k=n+1}^{m} u_k(x)\right|$$

$$\le \sum_{k=n+1}^{m} |u_k(x)|$$

$$\le \sum_{k=n+1}^{m} M_k < \varepsilon.$$

Here the series converges uniformly by the Cauchy criterion.

We now present our Main Theorem

Suppose that $\{S_n\}$ is a sequence of functions each of which is continuously differentiable on an interval $I = [a, b]$.

Suppose further that $\{S_n\}$ converges at one point x_0 in I and that $\{S_n'\}$ converges uniformly in I.

Then $\{S_n\}$ converges uniformly in I to a function and $S' = \lim S_n'$.

Proof: By the Fundamental Theorem of Calculus, for any $x \in I$, we have

$$S_n(x) = \int_{x_0}^{x} S_n'(t)\,dt + S_n(x_0).$$

Thus,

$$S_n(x) - S_m(x) = \int \left[S_n'(t) - S_m(t) \right] dt$$

$$+ \left[S_n(x_0) - S_m(x_0) \right]$$

$$\left| S_n(x) - S_m(x) \right|$$

$$\leq \int_{x_0}^{x} \left| S_n'(t) - S_m'(t) \right| dt$$

$$+ \left| S_n(x_0) - S_m(x_0) \right| .$$

Let $\varepsilon > 0$. Then the Cauchy

Criterion implies there is an

integer $N(\varepsilon)$ such that

if $m, n > N$, then

$$|S_n'(t) - S_m'(t)| < \varepsilon$$

and

$$|S_n(x_0) - S_m(x_0)| < \varepsilon .$$

Thus

$$|S_n(x) - S_m(x)| \leq \varepsilon (x - x_0) + \varepsilon$$

and so, $S_n(x)$ converges uniformly to a number $S(x)$ for each $x \in I$.

We denote the limit of S_n'

by σ (Thus $\sigma(t)$ is a function of t as is $S_n'(t)$)

It remains to show that

$S' = \sigma$, i.e., $S'(t) = \sigma(t)$.

We see that

$$S_n(x) - S_n(x_0) = \int_{x_0}^{x} \sigma(t)\,dt$$

By taking limits on both sides,

we get

$$S(x) - S(x_0) = \int_{x_0}^{x} \sigma(t)\,dt$$

By the Fundamental Theorem of calculus, we obtain

$$S'(x) = \sigma(x)$$

limit of derivative

derivative of limit.

On the other hand,

$$S(x) = \sum_{k=0}^{\infty} a_k x^k .$$

$$S'(x) = \left(\sum_{k=0}^{\infty} a_k x^k \right)',$$

$$\text{so } S'(x) = \sigma(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

so we have

$$S'(x) = \sigma(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Thus termwise differentiation is valid.

Application to power series.

Let $S_n(x) = \sum_{k=0}^{\infty} a_k x^k$

If we differentiate, we get

$$S_n'(x) = \sum_{k=1}^{n-1} a_k x^{k-1}$$

We define $\sigma = \lim S_n'$,

then $\sigma = \sum_{k=1}^{\infty} k a_k x^{k-1}$

Example:

$$f(x) = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}.$$

$$f'(x) = \frac{-(-2)}{(1-2x)^2} = \frac{2}{(1-2x)^2}$$

Term-by-term:

$$\sum_{k=1}^{\infty} 2^k k x^{k-1} = f'(x)$$

$$f(x) = \frac{1}{1-2x} = \sum_{k=1}^{\infty} 2^k k x^{k-1}$$