Absolute Value 2.2.

We can define $|a|$ as follows:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

We'll need these identities:

(a) $|-a| = |a|$

(b) $|ab| = |a||b|$

(c) $|a|^2 = a^2$

(d) $-|a| \leq a \leq |a|$

(e) if $b < 0$, then $|b| = -b$.

Proof.

(a) Suppose $a \geq 0$. Then $-a \leq 0$

$\rightarrow |-a| = -(-a) = a = |a|$

If $a < 0$, then $-a > 0$, so

$|-a| = -a = |a|$

by def. of $|a|$
when $a < 0$

(b) If either a or $b = 0$, then

both sides equal 0.

Now suppose $a, b > 0$.

$|ab| = ab = |a||b|$

since $ab > 0$

Now suppose $a > 0$, $b < 0$.

$|ab| = -ab = a(-b) = |a||b|$

When $a < 0$ and $b > 0$, and

$a, b < 0$, the argument is

similar.

(c) Since $a^2 \geq 0$,

$$a^2 = |a^2| = |a||a| = |a|^2.$$

(d). When $a \geq 0$, $a = |a|$

$\therefore -|a| \leq 0 \leq a \leq |a|$

Similarly, when $a \leq 0$,

$|a| = -a$, or $-|a| = a \leq 0 \leq |a|$

$-|a| = a \leq 0 \leq |a|$

Hence, $-|a| \leq a \leq |a|$

The following inequality is very useful.

Triangle Inequality.

If $a, b \in \mathbb{R}$, then

$|a+b| \leq |a| + |b|$.

Pf. Suppose first that $a+b \geq 0$

$\rightarrow |a+b| = a+b \leq |a| + |b|$

using (d)

Now suppose that $a+b<0$

$$\rightarrow |a+b| = -(a+b)$$

$$= -a-b \leq |a|+|b|$$

using (d).

which implies the Triangle Inequality. We can prove

$$|a-b| \leq |a|+|b| \qquad (1)$$

by replacing b by $-b$.

We will also need:

$$\big|\,|a| - |b|\,\big| \leq |a - b| . \qquad (+)$$

pf.

$$a = (a - b) + b$$

$$|a| \leq |a - b| + |b|$$

$$\rightarrow \left(|a| - |b|\right) \leq |a - b| \qquad (2)$$

Similarly $b = b - a + a$

$$|b| \leq |b - a| + |a|$$

$$|b| - |a| \leq |b - a|$$

$$-\left(|a| - |b|\right) \leq |a - b| \qquad (3)$$

By combining (2) and (3),

we obtain

$$\big|\, |a| - |b| \,\big| \leq |a - b| ,$$

which proves (*).

Another version is the

Backwards Triangle Property

$$|a-b| \geq |a| - |b|.$$

Pf.

$$|a| = |(a-b)+b|$$

$$\leq |a-b| + |b|$$

$$\Rightarrow |a-b| \geq |a| - |b|$$

One more inequality:

<u>Estimate</u>. Suppose that $c \geq 0$.

(1) $|a| \leq c$ if and only if

$$-c \leq a \leq c.$$

Let P and Q be statements.

Then P is true if and only if

Q is true, means that

P is true if Q is true

i.e., $Q \Rightarrow P$

and

P is true only if Q is true.

i.e., $P \Rightarrow Q$.

We prove (1) in 2 separate cases

Case 1: Suppose $a \geq 0$.

Since $|a| \leq c$, $\Rightarrow$ $a \leq c$

$\rightarrow$ $-c \leq 0 \leq a \leq c$,

$\rightarrow$ $-c \leq a \leq c$.

On the other hand. if

$-c \leq a \leq c$. then $|a| \leq c$

Case 2. Suppose $a < 0$.

If $|a| \leq c$, then $-a \leq c$

$\rightarrow$ $a \geq -c$.

Hence, $-c \le a < 0 \le c$

$\rightarrow -c \le a \le c.$

On the other hand, if

$-c \le a \le c$, then

$-a \le c \rightarrow |a| \le c.$

This proves (1) is true if

$a < 0.$

This proves the <u>estimate</u> in both cases.

We obtain $|a| \le c$

Thus, we've proved both

directions.

Ex. Find the set A of all x

such that $|3x+4| < 2$.

Set $c = 2$

$\therefore$ Left half is

and $a = 3X+4$.

$|a| < c \to -c < a < c$

or $-2 < 3X+4 < 2$

$\therefore \quad -6 < 3x < -2$

$\longrightarrow \quad -2 < x < -\frac{2}{3}.$

Ex. Set $f(x) = \dfrac{2x^2 - 4x + 3}{5x - 2}.$

when $1 \le x \le 2$. Estimate $|f(x)|$.

For the numerator;

$$|2x^2 - 4x + 3| \le |2x^2| + |4x| + 3$$

$$\le 8 + 8 + 3 = 19$$

For the denominator:

$$|5x-2| \geq |5x| - |2|$$

$$\geq 5 - 2 = 3$$

Hence,

$$|f(x)| \leq \frac{19}{3}$$

Def'n. Let $a \in \mathbb{R}$ and $\varepsilon > 0$.

Then the ε-neighborhood of

a is the set

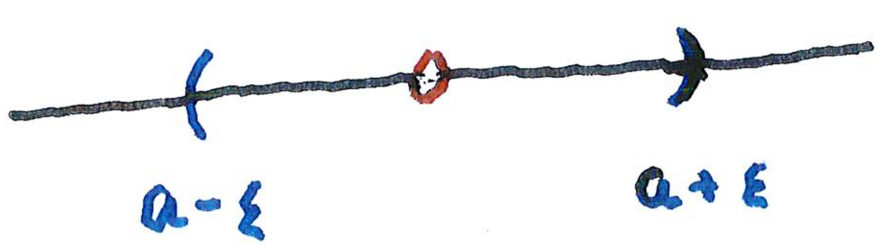
$$V_\varepsilon(a) = \{ x \in \mathbb{R} : |x-a| < \varepsilon \}.$$

If we replace a in (1) by $x-a$ and c by ε, it follows that $x \in V_\varepsilon(a)$ if only if

$$-\varepsilon < x-a < \varepsilon$$

or $$a-\varepsilon < x < a+\varepsilon$$

On the real line this is



Thm. Let $a \in \mathbb{R}$. If x belongs to $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Pf. Suppose $x \neq a$. If we set $\varepsilon = \frac{|x-a|}{2}$ in the definition of $V_\varepsilon(a)$, then

$$|x-a| < \frac{|x-a|}{2}.$$

Dividing by $|x-a|$, we have $1 < \frac{1}{2}$. This contradiction $\rightarrow$ $x = a$.