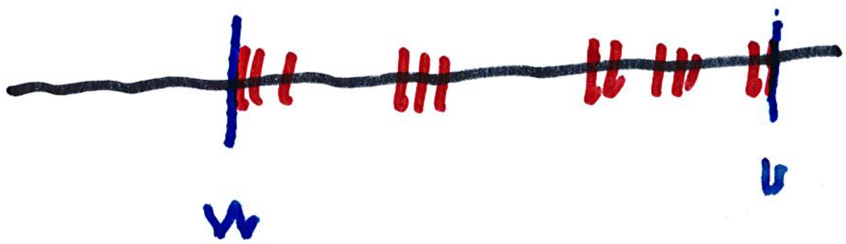
2.3 The Completeness Property

In this section we show that a bounded subset S of $\mathbb{R}$ has a "maximum" u and a "minimum" w.



We say that S is bounded above if there is a number u

such that $s \leq u$ for all $s \in S$.

Each such number u is called an <u>upper bound</u> of S

Similarly, we say S is bounded below if there is a number w such that $w \leq s$ for all $s \in S$.

Each such number w is called a lower bound of S.

Example. $S = \{x \in \mathbb{R} ; x < 2\}$ is bounded above but not bounded below.

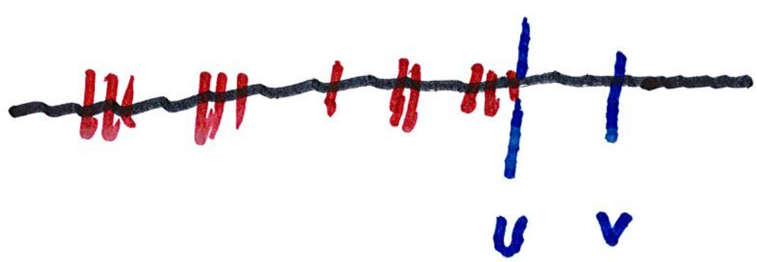
Definition. The number u is a supremum of S (also written as $\sup S$ or least upper bound of S)

if

(1′) u is an upper bound of S and

(2′) if v is any upper bound of S

then $v \geq u$



Similarly, w is an infimum of S

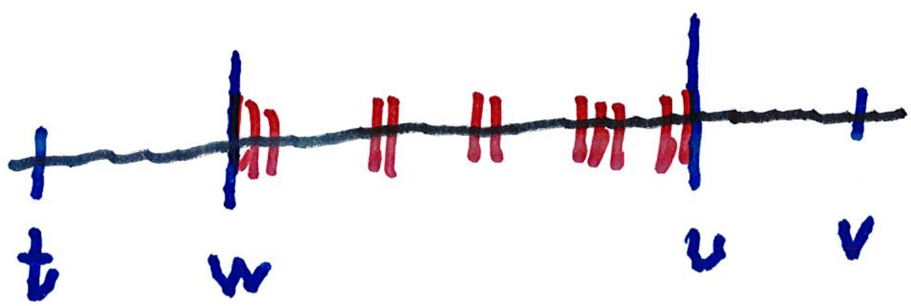
if

(1') w is a lower bound of S

and

(2') if t is any lower bound of S,

then $t \leq w$



Thus $u = \sup S$, and

$w = \inf S$.

One can show there can only be one supremum of S and one infimum of S.

Suppose there 2 numbers u_1 and u_2 that are both suprema of S. The fact that

$u_2 = \sup S$ and u_1 is an upper bound of S implies that $u_1 \geq u_2$. The same reasoning implies that $u_2 \geq u_1$.

It follows that $u_1 = u_2$.

Given that u is an upper bound of S, we can express the fact $u = \sup S$ in 4 ways that are all equivalent

(1) If v is an upper bound of S, then $v \geq u$.

(2) If $z < u$, then z is <u>not</u> an upper bound of S

$1 \to 2$ (For if z were an upper bound, then it would satisfy $z < u$, which contradicts (1))

(3) If $z < u$, then there must be an $s_z \in S$ that satisfies $s_z > z$.

$2 \to 3$

For if $s \leq z$ for all $s \in S$, this would imply that z <u>is</u> an upper bound, which contradicts (2). Hence there is $s_z \in S$ with $s_z > z$.

(4) For every $\varepsilon > 0$, there is an $s_\varepsilon \in S$ with $s_\varepsilon > u - \varepsilon$.

$3 \to 4$

Just set $z = u - \varepsilon$ and note that $z < u$. By (3), there is a number s (which we write as s_ε) such that $s_\varepsilon > u - \varepsilon$. This proves (4).

All that remains is to show that (4) implies (1)

First, suppose that x is a number such that

$x < \varepsilon$, for all $\varepsilon > 0$.

Then $x \leq 0$. It suffices to assume that $x > 0$.

If we set $\varepsilon = x$, then we obtain

$x < x$, which is impossible.

Thus, $x \leq 0$.

$4 \to 1$

Now let $\varepsilon > 0$. Then (4) implies

that there is $s_\varepsilon \in S$ so that

$s_\varepsilon > u - \varepsilon$. Let v be any

upper bound of S. Then

$v \geq s_\varepsilon > u - \varepsilon$, or

$u - v < \varepsilon$, for all $\varepsilon > 0$.

It follows from the above

argument that $u - v \leq 0$

$\to$ $v \geq u$. This proves (1).

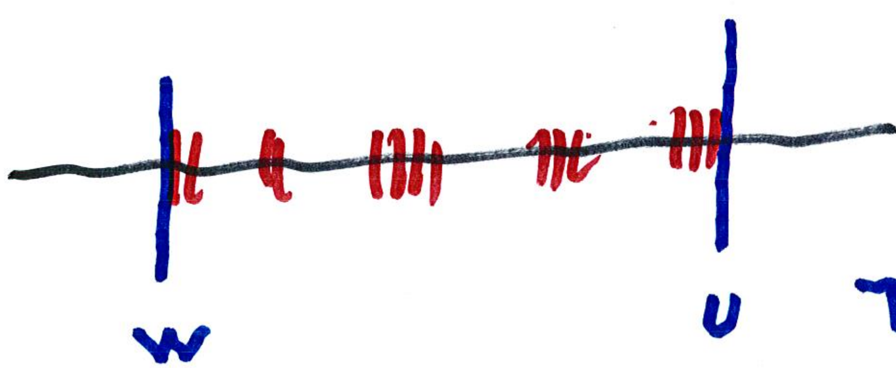
One can show from the construction of $\mathbb{R}$, that the following is true:

Completeness Property of $\mathbb{R}$.

(α) If S is any subset of $\mathbb{R}$ that if S is bounded above, then there is a number u such that $u = \sup S$.

Similarly

(B) If S is any subset of $\mathbb{R}$ that is bounded below then there is a number w such that $w = \inf S$



This set S is bounded.

Example. Let $S = [a, b)$.

i.e. $a \leq s < b$. (1)

We first show that $\sup S = b$.

Since $s < b$, it follows that

b = an upper bound of S.

Let $v \in [a, b)$. Set $s = \frac{v+b}{2}$.

This implies $v < s$. Therefore

$v \neq$ an upper bound of S.

Now let $v < a$. If we set

$s = a$. Then $v < s$. Then

v is not an upper bound of S.

Thus, if $v < b$, then v is

NOT an upper bound of S

Hence, if v is an upper bound,

then $v \geq b$. It follows that

$\sup S = b$.

Now we show that $\inf S = a$.

Note that (1) implies that

a is a lower bound of S

Now suppose that t is

any lower bound of S. Then

$t \le s$, for all $s \in S$.

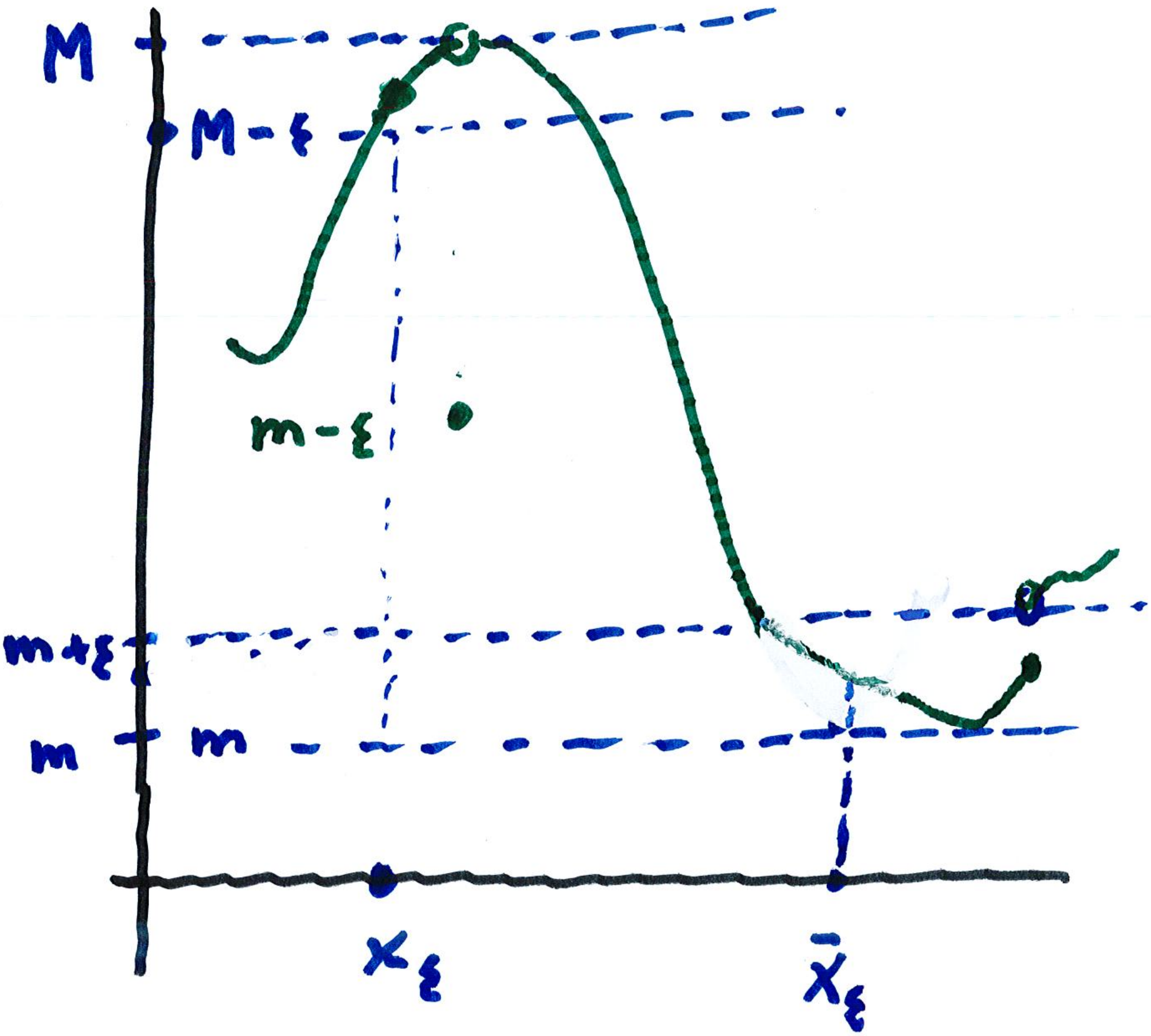
In particular, if we set $s = a$,

we get $t \le a$ Hence $\inf S = a$

Ex. Let f be a function on an interval I such that there is a constant A such that $|f(x)| \leq A$, for all $x \in I$.

Note that f is bounded above by A and bounded below by $-A$. Set $S = \{ f(x) : x \in I \}$

Set $M = \sup S$ and $m = \inf S$



By definition, M is an upper bound, so $f(x) \le M$, for $x \in I$

Also m is a lower bound, so

$f(x) \geq m$, for all $x \in I$.

For any $\varepsilon > 0$, there is a point $\bar{x}_\varepsilon \in I$, so that

$f(\bar{x}_\varepsilon) > m + \varepsilon$.