Applications of Completeness

Archimedean Property.

1. If $x > 0$, then there exists $n_x \in \mathbb{N}$ so that $x < n_x$.

<u>Pf.</u> Suppose this is NOT true. Then for every $n \in \mathbb{N}$, we would have $n \le x$, for all n in $\mathbb{N}$. By the Completeness Property, $\mathbb{N}$ has a supremum u.

Then $u-1$ is not an upper bound of $\mathbb{N}$, so there is an integer $m \in \mathbb{N}$ with $u-1 < m$. Adding 1, we get $u < m+1$. This contradicts the fact that $n \leq u$ for all n. Hence, there is an integer n_x with $n_x > x$.

2. For any $\varepsilon > 0$, there is an integer K in $\mathbb{N}$ so that $\frac{1}{n} < \varepsilon$, for all $n \geq K$.

Pf. Set $x = \frac{1}{\varepsilon}$. We showed above that there is an integer n_x, such that $n_x > x$. If we set $K = n_x$, and if $n \geq K$, then

$$n \geq n_x > x = \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon.$$

3. If $y > 0$, then there exists $n_y \in \mathbb{N}$ such that

$$n_y - 1 \leq y \leq n_y \qquad (*)$$

Pf. The Archimedean Property implies that the subset $E_y = \{m \in \mathbb{N} : y < m\}$ is nonempty. The Well-Ordering Property implies any nonempty subset $E \subset \mathbb{N}$

has a least element. Thus

E_y has a least element,

which
we denote by n_y. Then

$n_y - 1$ does not belong to E_y

Hence we have

$n_y - 1 \leq y < n_y$

Density Theorem.

If x and y are any real numbers with $x < y$, then there is a rational number $r \in \mathbb{Q}$ such that $x < r < y$

Pf. We can assume that $x > 0$. (Let $m \in \mathbb{N}$ satisfy $m + x > 0$. Then replace x with $x+m$ and y with $y+m$)

Since $y-x > 0$, it follows from 2. that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y-x$.

which gives $nx+1 < ny$. (i)

If we apply (*) to nx, we obtain $m \in \mathbb{N}$ with

$$m-1 \leq nx < m.$$

Therefore,

$$m \leq nx+1 \underset{\text{by (i)}}{<} ny.$$

which leads to

$$nx < m < ny.$$

Thus the rational number

$r = m/n$ satisfies

$$x < r < y$$

2.4. Applications of Least Upper Bound Property.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence.

1. We say $\{x_n\}$ is increasing if $x_{n+1} \geq x_n$, for all $n = 1, 2, \ldots$

2. We say $\lim_{n \to \infty} x_n = \tilde{x}$ if for all $\varepsilon > 0$, there is an

integer $N_\varepsilon > 0$ so that if

$n \geq N_\varepsilon$, then

$$|x_n - \tilde{x}| < \varepsilon, \quad \text{for all } n \geq N_\varepsilon.$$

Monotone Convergence Thm

Suppose $\{x_n\}$ is an

increasing sequence such that

$x_n \leq M$, for all $n = 1, 2, \ldots$.

Then there is a number

$\tilde{x} \leq M$, such that

$\lim_{n\to\infty} x_n = \tilde{x}$.

Pf. Let $S = \{ x_n ; n = 1, 2, \dots \}$

and let $\tilde{x} = \text{l.u.b } S$.

Choose $\varepsilon > 0$. Then

there is an integer $N_\varepsilon > 0$

so that $x_{N_\varepsilon} > \tilde{x} - \varepsilon$.

Since $\{ x_n \}$ is increasing,

if $n \geq N_\varepsilon$, then

$$\tilde{x} - \varepsilon < x_{N_\varepsilon} \leq x_n \leq \tilde{x}.$$

The last inequality follows from the fact that

$$x_n \leq \tilde{x} = \text{l.u.b. } S.$$

Hence $\tilde{x} - \varepsilon < x_n \leq \tilde{x} < \tilde{x} + \varepsilon$

i.e., $-\varepsilon < x_n - \tilde{x} < \varepsilon$

for $n \geq N_\varepsilon$.

$$\therefore \lim_{n \to \infty} x_n = \tilde{x}.$$

Example. Suppose that f is a bounded function on an interval I. Then there is a number $A > 0$ so that

$$|f(x)| < A \quad \text{for all } x \in I, \text{ i.e.,}$$

$$-A < f(x) < A.$$

If we let $S = \{f(x) ; x \in I\}$

Then S has an infimum

$m_1 = \inf S$, and S has a supremum $m_2 = \sup S$.

We conclude that

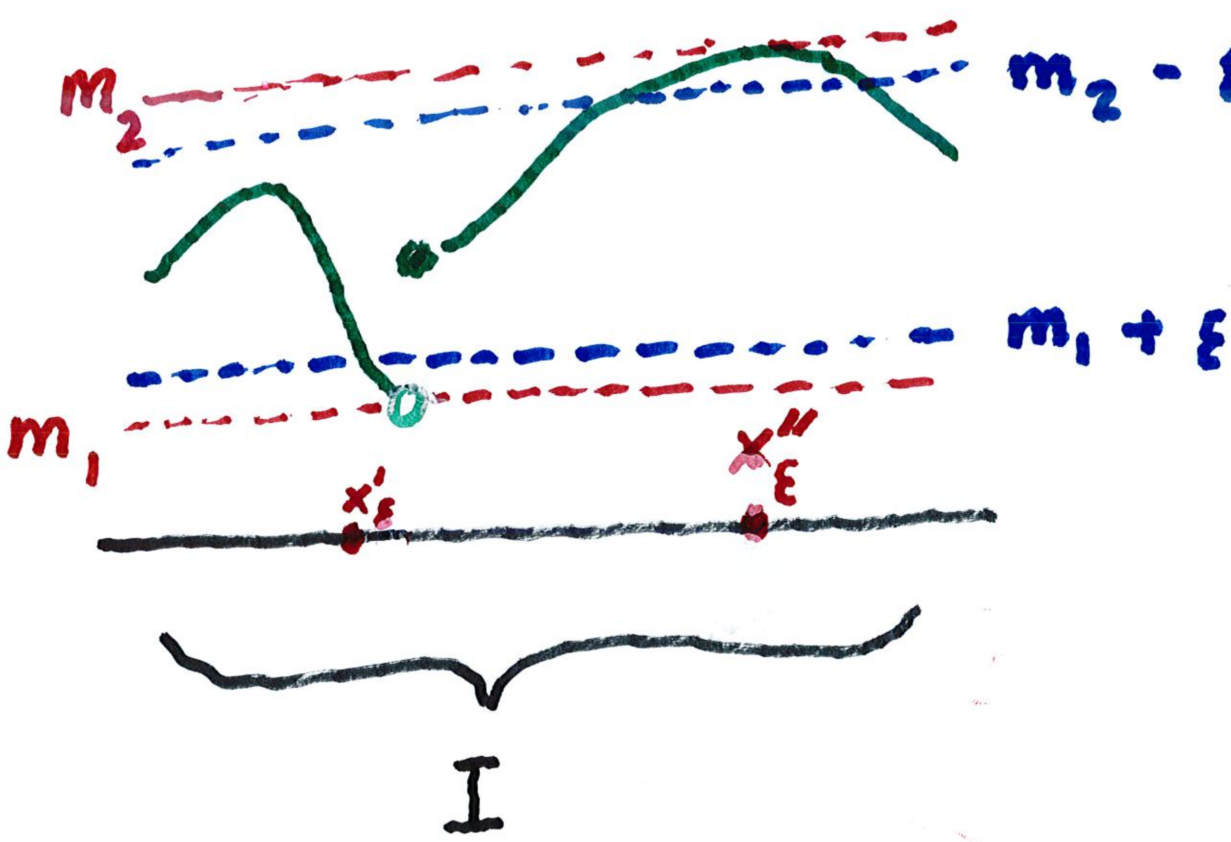
$m_1 \leq f(x)$ for all $x \in I$, and for every $\varepsilon > 0$, there is an x'_ε so that

$$m_1 \leq f(x'_\varepsilon) < m_1 + \varepsilon.$$

Also, since $m_2 = \sup S$

for every $\varepsilon > 0$, there is an x''_ε so that

$$m_2 - \varepsilon < f(x''_\varepsilon) \leq m_2.$$



Problem 2.4.2.

Let $S = \left\{ \frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{I} \right\}$

Calculate $\sup S$ and $\inf S$.

Note first:

using $\frac{1}{n} \leq 1$

$$\frac{1}{n} - \frac{1}{m} < 1 - 0 = 1$$

and $\frac{1}{m} > 0$

$$\frac{1}{n} - \frac{1}{m} > 0 - 1 = -1.$$

using $-\frac{1}{m} \geq -1$

and $\frac{1}{n} > 0$

It seems likely that

$\sup S = 1$ and $\inf S = -1$

Note that $1 =$ an upper bound of S, and $-1 =$ a lower bound.

Set $m = 1$ and, for every $\varepsilon > 0$, there is an n_ε, so that $\frac{1}{n_\varepsilon} < \varepsilon$

Then $\frac{1}{n_\varepsilon} - \frac{1}{m} < \varepsilon - 1$.

Thus $\inf S = -1$.

Similarly, set $n = 1$ and choose $m_\varepsilon \in \mathbb{N}$ so that

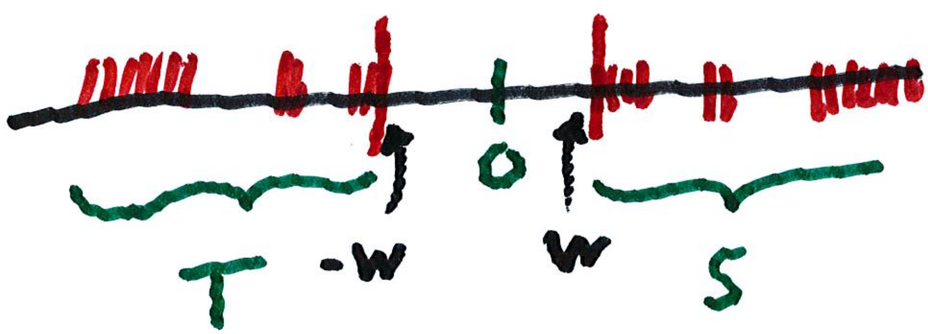
$\frac{1}{m_\varepsilon} < \varepsilon$. Then

$$\frac{1}{n} - \frac{1}{m_\varepsilon} < 1 - \varepsilon.$$

It follows that $\sup S = 1$.

Ex. Let S be a subset of $\mathbb{R}$ that is bounded below.

Let $T = \{-x : x \in S\}$



Then $\inf S = \sup\{-x : x \in S\}$

Pf: Let $w = \inf S$. Then by Criterion 4 on p. 38, a number w is an infimum of S

if (i) w is a lower bound,

and if (ii) for every $\varepsilon > 0$,

there is a $y_\varepsilon \in S$ such that

$$y_\varepsilon < w + \varepsilon.$$

Since w is a lower bound

of S, it satisfies $w \le x$,

for all $x \in S$.

Multiplying by (-1), we get

$-w \geq -x$, for all $x \in S$.

Since every element of T

is given by $-x$, for $x \in S$,

we conclude that $-w$

is an upper bound of T.

Let $\varepsilon > 0$, then

$-y_\varepsilon > -w - \varepsilon$. Again

(Note that $-y_\varepsilon \in T$)

Criterion 4 on p. 38

implies that $-w = \sup T$

$$= \sup\{-x : x \in S\}$$

We conclude that $-w =$

$$\inf S = w = -(-w) = -\sup T$$

$$= -\sup\{-x : x \in S\}$$

This gives the desired equality.