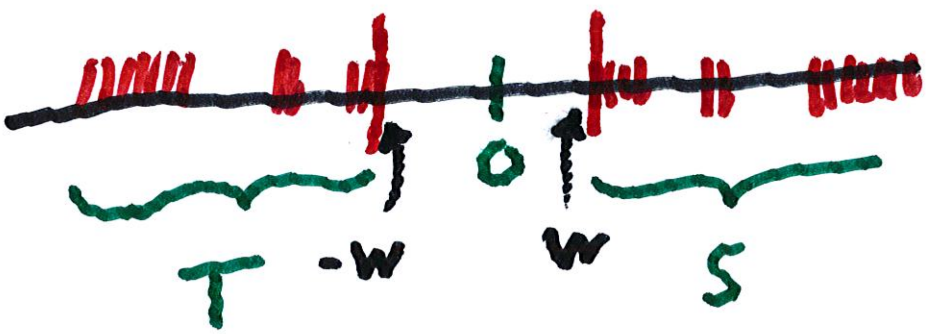
Ex. Let S be a subset of $\mathbb{R}$ that is bounded below.

Let $T = \{-x : x \in S\}$



Then $\inf S = \sup\{-x : x \in S\}$

Pf: Let $w = \inf S$. Then by Criterion 4 on p. 38, a number w is an infimum of S

if (i) w is a lower bound,

and if (ii) for every $\varepsilon > 0$,

there is a $y_\varepsilon \in S$ such that

$y_\varepsilon < w + \varepsilon$.

Since w is a lower bound

of S, it satisfies $w \le x$,

for all $x \in S$.

Multiplying by (-1), we get

$-w \geq -x$, for all $x \in S$.

Since every element of T is given by $-x$, for $x \in S$, we conclude that $-w$ is an upper bound of T.

Let $\varepsilon > 0$, then

$-y_\varepsilon > -w - \varepsilon$. Again

(Note that $-y_\varepsilon \in T$)

Criterion 4 on p. 38

implies that $-w = \sup T$

$= \sup\{-x : x \in S\}$

We conclude that $-w =$

$\inf S = w = -(-w) = -\sup T$

$= -\sup\{-x : x \in S\}$

This gives the desired equality.

2.5 Intervals

We need to prove a theorem

about "nested intervals"

before we study 3.4.

We say a sequence of closed

$\underbrace{\text{bounded}}_{}$ intervals are **nested** if

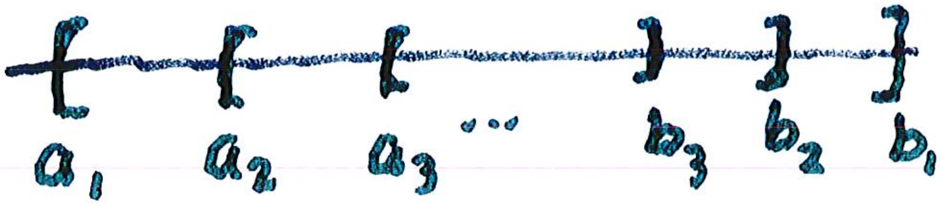
$I_1 \supseteq I_2 \supseteq \dots \supset I_n \supset I_{n+1} \supset \dots$

If $I_n = [a_n, b_n]$, then

(b_n) is decreasing, and

(a_n) is increasing, i.e.

we have the picture



We prove the

Nested Interval Property:

Given a sequence of nested closed intervals as above, there is a point η in I_n for all $n \in N$

Proof. Since $I_n \subseteq I_1$,

we get

$a_n \leq b_n \leq b_1$, for all $n \in \mathbb{N}$.

Hence the sequence (a_n)

is increasing and bounded.

By the Monotone Convergence

Thm., there is an η

satisfying $\eta = \lim(a_n)$.

Clearly $a_n \le \eta$, all $n \in \mathbb{N}$. (1)

We want to show that

$\eta \le b_n$ for all n.

We do this by showing that

for any particular n,

$b_n \ge a_k$, $k = 1, 2, \dots$

There are 2 cases.

(i) If $n \le k$, then since

$I_n \supseteq I_k$, we have

$a_k \leq b_k \leq b_n.$

(iii) If $k < n$, then since

$I_k \supseteq I_n$, we have

$a_k \leq a_n \leq b_n$

We conclude that $a_k \leq b_n$.
for all k,

so that b_n is
an upper bound for
$\{a_k ; k \in N\}$

It follows that

$b_n \geq \eta$, for all $n \in N$,

which implies that

$a_n \leq \eta \leq b_n$, for all $n \in N$,

which inturn implies that

$\eta \in I_n$ for all $n \in N$.

We can use nested intervals to show that the set $\mathbb{R}$ of real numbers is NOT countable.

Suppose that there is a sequence $I = \{x_1, x_2, \ldots\}$ such that for any x in $[0,1]$, there is an integer n such that $x_n = x$.

Choose a closed subinterval

$I_1 \subset [0,1]$ such that $x_1 \notin I_1$.

Now choose a closed subinterval

$I_2 \subset I_1$ such that $x_2 \notin I_2$.

In this way we obtain

a sequence of closed subintervals

such that

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n$$

such that for all $n = 1, 2, \ldots,$

$$x_n \notin I_n \qquad \underbrace{[\ \cdot\ \underbrace{[\qquad]}_{I_n}\ ]}_{I_{n-1}}$$

The Nested Interval Theorem

implies that there is a

point $\eta \in I_n$, for all

$n = 1, 2, \ldots$

Since $x_n \notin I_n$ for all n,

it follows that

for all $n = 1, 2, \ldots,$

$x_n \neq \eta$.

It follows that $I = [0, 1]$

is <u>not</u> countable.

Decimal Expansions.

Given any number $x \in [0,1]$,

we can write a decimal

expansion as

$$\frac{b_1}{10^1} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} \qquad (1)$$

$$\leq x \leq \frac{b_1}{10^1} + \frac{b_2}{10^2} + \cdots + \frac{(b_n+1)}{10^n}.$$

where $b_j \in \{0, 1, \ldots, 9\}$

For example, if $x = .46329\ldots$

then $x \in I$, and

$$\frac{4}{10^1} < x < \frac{5}{10^1} .$$

Then we subdivide $\left[\frac{4}{10}, \frac{5}{10}\right] = I_1$

into ten intervals of length $\frac{1}{10^2}$

$$\frac{4}{10^1} + \frac{6}{10^2} < x < \frac{4}{10^1} + \frac{7}{10^2} ,$$

The 7-th such interval of

length $\frac{1}{10^2}$ is

$$\left[\frac{4}{10^1} + \frac{6}{10^2}, \frac{4}{10^1} + \frac{7}{10^2}\right]$$

which we can write as

$$[.46, .47].$$

If (1) holds for all n,

then x has a decimal expansion

given by $x = .b_1 b_2 b_3 \dots b_n \dots$

If $x \geq 1$, and if $B \in N$

is such that $B \leq x < B+1$,

then $x = B.b_1 b_2 b_3 \dots b_n \dots$

where the decimal

representation of $x - B \in [0,1)$

is as above.

In this way, we can

define all nonnegative

real numbers.