1.2 Principle of Mathematical Induction

Let S be a subset of $\mathbb{N}$ that satisfies:

(1) The number $1 \in S$

and

(2) For every $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$.

Then for all $n \in \mathbb{N}$, $n \in S$.

Note (2) does not ask us to prove that $k \in S$.

We only need to show that "if $k \in S$, then $k+1 \in S$".

Usually, Math. Ind. is used to prove that a sequence of statements are all true.

For each $n \in \mathbb{N}$, let $P(n)$ be a meaningful statement about $n \in \mathbb{N}$. We let

$$S = \{ n \in \mathbb{N} ; P(n) \text{ is true.} \}$$

The above Mathematal Induction Principle becomes:

Suppose that

(1') $P(1)$ is true.

(2') For every $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Ex. Suppose $P(n)$ is the statement that

$f(n) = n^2 - n + 41$ is prime.

Note that when $n = 1$,

$1^2 - 1 + 41$ is prime.

Then $P(1)$ is true.

But (after some calculation)

$f(40) = 1601$ is prime and

$f(41) = 41^2 = 1681$

$\therefore$ $f(41)$ is NOT prime.

Hence $P(40)$ is true but

$P(41)$ is false.

Thus (2) fails when $n = 40$

Ex. Use Math. Ind. to prove

that

$$1^2 + 2^2 + 3^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$$

When $n=1$, $P(1)$ is the statement

$$1^2 = \frac{1\cdot 2\cdot(3)}{6} = 1$$

$\therefore$ $P(1)$ holds.

Now suppose $P(k)$ is true.

Then

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

by the induction assumption (2').

Now check $P(k+1)$:

$$1^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + \frac{(k+1)}{6} \right]$$

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6k + 6 \right]$$

$$= \frac{(k+1)}{6} \left[2k^2 + 7k + 6 \right]$$

$$= \frac{(k+1)}{6} (k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$\therefore$ P(k+1) is true.

$\therefore$ (2′) holds

Since both (1') and (2')

are true, it follows that

$P(n)$ is true for all $n \in \mathbb{N}$.

Hence

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Ex. Prove that $5^{2n} - 1$ is divisible by 8. This is $P(n)$.

(1) When $n=1$,

$5^2 - 1 = 24 = 3 \cdot 8$,

so $P(1)$ is true.

(2) Suppose that $P(k)$ is true, i.e., $5^{2k} - 1$ is divisible by 8.

Check $P(k+1)$:

$$5^{2(k+1)} - 1 = 5^2 \cdot 5^{2k} - 1$$

$$= 5^2 (5^{2k} - 1) + 5^2 - 1$$

$$= 5^2 (5^{2k} - 1) + (5^2 - 1)$$

Both are divisible by 8

$\therefore$ $P(k+1)$ is true.

$\rightarrow$ (2) holds $\Rightarrow$ $5^{2n} - 1$ is div. by 8 for all n.

Bernoulli's Inequality

Show that

for all $n \in \mathbb{N}$ and for all $x > -1$,

$$(1+x)^n \geq (1+nx)$$

Pf. First we check $P(1)$

$$(1+x)^1 = (1+1\cdot x) \quad \checkmark.$$

Now check (2)

Suppose that $(1+x)^k \geq 1+kx$

for all $x > -1$.

Note that

$$(1+x)^{(k+1)} = (1+x)^k (1+x)$$

$$\geq (1+kx)(1+x)$$

by the inductive hypothesis

and that $1+x > 0$.

$$= 1 + kx + x + kx^2$$

$$\geq 1 + (k+1)x .$$

Thus $P(k+1)$ is true,

and hence (2) holds.

By induction, $P(n)$ is true

for all $n \in \mathbb{N}$

$$\Rightarrow (1+x)^n \geq 1 + nx, \text{ when } x > -1.$$

Sometimes, the statement is only defined for $n \geq n_0$

Modified Principle of Math. Induction.

Suppose that

(1) $P(n_0)$ is true.

(2) For all $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$

Ex. Prove that

$2^n < n!$ for all $n \geq 4$.

Note that when $n = 4$,

$$2^4 = 16 < 24 = 4!$$

This shows that $P(4)$ holds.

Now let k be an integer

≥ 4, and assume that

$$2^k < k!$$

Note that since $k \geq 4$

$$2^{(k+1)} = 2^k \cdot 2 < (k!)2$$

$$< (k!)(k+1) = (k+1)!$$

↑

Since $2 < k+1$.

Hence $P(k+1)$ is true. By induction $P(n)$ is true for all $n \geq 4$.

Sometimes the Induction Principle can be expressed as follows.

Let S be a subset of $\mathbb{N}$ such that

(1) $P(1)$ is true.

(2) For every $k \in \mathbb{N}$,

if $P(1), \ldots, P(k)$ are all true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

(This is sometimes called the Principle of Strong Induction.)

Ex. Suppose a sequence

$\{x_n\}$ is defined by

$x_1 = 1, \quad x_2 = 2$ and

$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$.

Use Strong Induction to

show that

$1 \leq x_n \leq 2$, all $n \in \mathbb{N}$.

Let $P(n)$ be the statement

that $1 \le x_n \le 2$.

Note that $P(1)$ and $P(2)$

both hold by hypothesis.

Now let $k \in \mathbb{N}$ with $k \ge 2$,

and suppose that $P(j)$ is

true for all $j \le k$, i.e.,

$1 \le x_j \le 2$, if $1 \le j \le k$.

Then $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\leq \frac{1}{2}(2+2) = 2$

by strong induction hypothesis

and

$x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\geq \frac{1}{2}(1+1) = 1$

by strong ind. hypothesis

Hence $1 \leq x_{k+1} \leq 2$,

which shows that $P(k+1)$ is true. Thus the Strong Induction Principle implies that $P(n)$ is true for all $n \in N$.

Conjecture a formula for

$1+3+\cdots(2n-1)$.

Set $S_n =$

$n=1$, $S_1 = 1$

$n=2$, $S_2 = 1+3 = 4$

$n=3$, $S_3 = 1+3+5 = 9$

$n=4$, $S_4 = 1+3+5+7 = 16$

It seems that $S_n = n^2$.

Prove by induction.

When $n=1$, it is obvious

that $S_1 = 1 = 1^2$.

Now assume that $S_n = n^2$

$$S_{n+1} = \{1+3+\cdots+(2n-1)\} + \{2n+1\}$$

$$= n^2 + (2n+1)$$

$$= (n+1)^2.$$

This shows that (2) is true

for $n+1$.

By induction,

$$1+3+5+\dots+(2n-1) = n^2$$

for all $n \in \mathbb{N}$.

Ex. Find all $n \in \mathbb{N}$ such that

$n^2 < 2^n$.

Try $n=1$ $\quad 1 < 2^1$ ✓

$n=2$ $\quad 4 < 4$ ✗

$n=3$ $\quad 9 < 8$ ✗

$n=4 \quad 16 < 2^4 = 16 \quad \times$

$n=5 \quad 25 < 2^5 = 32 \quad \checkmark$

We prove $n^2 < 2^n$ for $n \geq 5$.

We already proved this if $n=5$.

Assume $n^2 < 2^n$ for $n \geq 5$

$$(n+1)^2 = \frac{(n+1)^2}{n^2} \cdot n^2 \leq \left(\frac{n+1}{n}\right)^2 2^n$$

inductive hypothesis.

If $n \geq 5$,

$$\left(1+\frac{1}{n}\right) \leq 1+\frac{1}{5} \leq \frac{6}{5}.$$

$$\therefore \left(\frac{n+1}{n}\right)^2 \le \left(\frac{6}{5}\right)^2 \le \frac{36}{25} < 2.$$

Hence

$$\left(\frac{n+1}{n}\right)^2 2^n < 2 \cdot 2^n = 2^{n+1}. \checkmark$$

We conclude by induction

that $n^2 < 2^n$ if

$n = 1$ or if $n \geq 5$