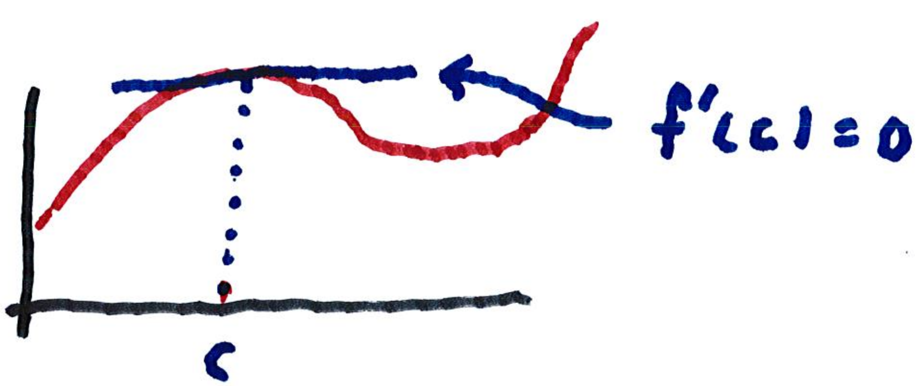
6.2 The Mean Value Theorem

Let $f: I \to \mathbb{R}$, where I is an interval. The function f has a relative maximum (or minimum) at $c \in I$ if there is a neighborhood $V_\varepsilon(c) = V$ of c such that $f(x) \leq f(c)$ (or $f(x) \geq f(c)$) for all x in V.



Interior Extremum Theorem.

Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$

Pf. We prove the theorem in the case when f has a relative maximum.

If $f'(c) > 0$, then there is a neighborhood $V \subseteq I$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ for all } x \in V, \text{ with } x \neq c.$$

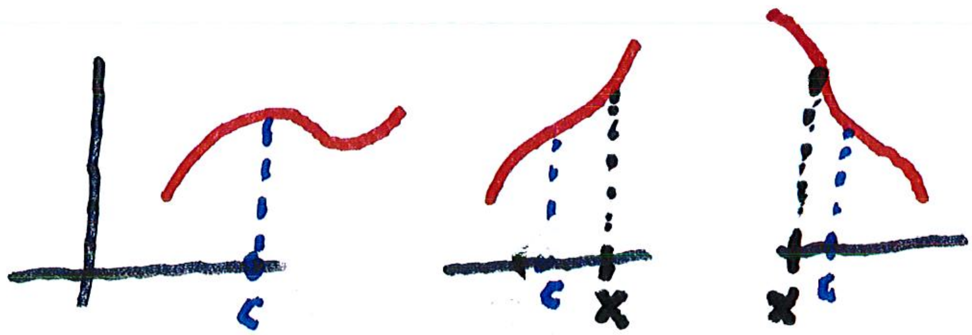
If $x \in V$ and $x > c$, then

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} > 0$$

This contradicts the hypothesis that f has a relative maximum at c.

Similarly, we cannot have

$f'(c) < 0$.



For if $f'(c) < 0$, then

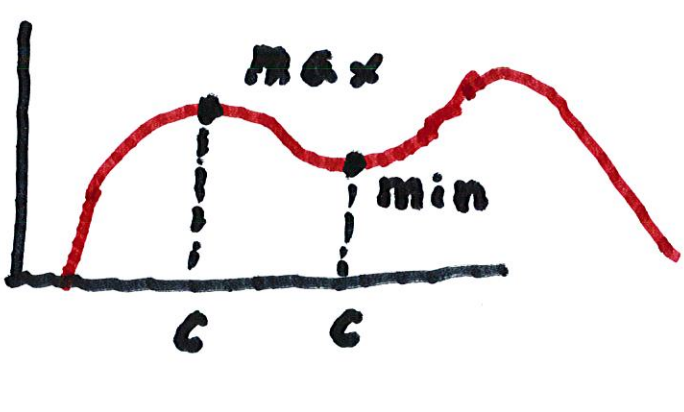
$$\frac{f(x) - f(c)}{x - c} < 0, \quad \text{all } x \in V, \; x \neq c$$

If $x \in V$ and $x < c$, then

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

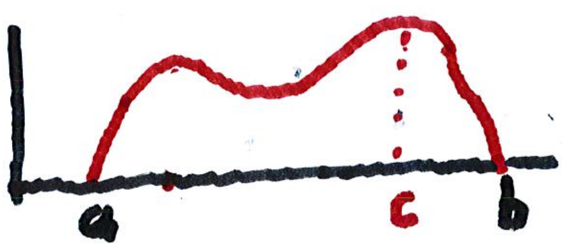
Rolle's Theorem. Suppose that $f: I \to \mathbb{R}$ is continuous on a closed interval $I = [a, b]$, that f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$.

Then there is at least one point c in (a, b) such that $f'(c) = 0$.



Proof. If $f(x)=0$ for all c in (a,b), then any point c satisfies the conclusion of the theorem. Thus, we can assume that f does not vanish identically. Replacing f by $-f$ if necessary, we can assume that f assumes some positive values. By the

Maximum-Minimum Thm, the function f attains the value $\sup\{f(x) : x \in I\}$ at some c in (a, b). Since $f(a) = f(b) = 0$, the point c must lie in (a, b).



Since f has a relative maximum at c, we conclude from the Interior Extremum Theorem

that $f'(c) = 0$.

We now prove the

<u>Mean Value Thm.</u> Suppose that

f is continuous on a closed

interval $[a, b]$, and that

f has a derivative in (a, b).

Then there is a point c in (a, b)

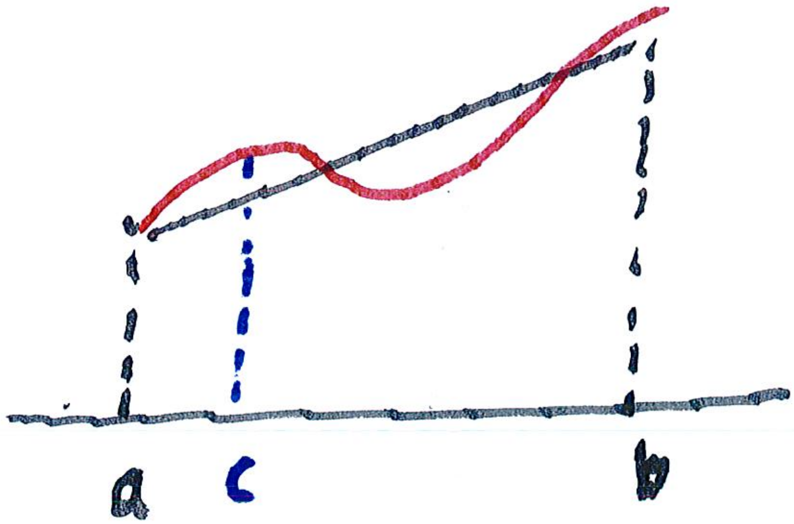
such that

$$f(b) - f(a) = f'(c)(b-a)$$

Pf. Consider the function

$$\varphi(x) = f(x) - f(a) - \frac{f(b-a)}{b-a}(x-a).$$

(The function φ is the difference between f and the function whose graph is the line segment joining $(a, f(a))$ and $(b, f(b))$



Note that $\varphi(a) = 0$ and $\varphi(b) = 0$. We can apply Rolle's Thm, which implies that there is a point $c \in (a,b)$ such that $\varphi'(c) = 0$. Hence

$$0 = \varphi'(c) = f'(c) - \frac{f(b) - f(a)}{b-a}$$

It follows that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Thm. Suppose that f is continuous on $[a,b]$, that f is differentiable on (a,b) and that $f'(x) = 0$ for all $x \in (a,b)$. Then f is a constant on $[a,b]$

Pf. We will show that

$f(x) = f(a)$ for all ~~$x \in [a,b]$~~.

$x \in [a,b]$. In fact,

if $x > a$, we apply the

Mean Value Theorem to

f on the closed interval

$[a,x]$. We obtain a

number c (dependent on x)

between a and x so that

$$f(x) - f(a) = f'(c)(x-a).$$

(by the MVT)

Since $f'(c) = 0$, we deduce

that $f(x) - f(a) = 0$.

Corollary: Suppose that

f and g are continuous

on $[a,b]$, that they are

differentiable on (a,b) and

that $f'(x) = g'(x)$, for all $x \in [a,b]$

Then there is a constant C so that $f = g + C$.

Pf. Just apply the above theorem to $f - g$.

We say that $f: I \to \mathbb{R}$ is increasing on I if whenever $x_1, x_2 \in A$ with $x_1 < x_2$, then $f(x_1) \leq f(x_2)$.

Also f is decreasing if $-f$ is increasing.

Thm. Let $f: I \to \mathbb{R}$ be differentiable on I. Then

(a) f is increasing if and only if $f'(x) \geq 0$, all $x \in I$.

Pf. (a) Suppose that $f'(x) \geq 0$ for all $x \in I$. If x_1, x_2 in I satisfy $x_1 < x_2$, then the Mean Value Thm (applied to f on $[x_1, x_2]$) implies that there is a point $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) \geq 0$, we conclude that $f(x_2) - f(x_1) \geq 0$. Hence f is increasing on I.

Now assume that f is increasing on I, and differentiable an I. Then

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

Passing to the limit, we obtain that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

6.3 L'hospital's Rules

Suppose that f, g are functions defined near c and that

$A = \lim_{x \to c} f(x)$ and

$B = \lim_{x \to c} g(x)$.

If $B \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

If $A = 0$ and $B = 0$, then

the situation is more

complicated. L'Hospital's

Rules handle this situation.

We will need a generalization of the Mean Value Theorem.

<u>Cauchy Mean Value Theorem.</u>

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ for all x in (a,b). Then there is a point c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Pf. Note that the hypothesis $g'(c) \neq 0$ implies that $g(b) \neq g(a)$ (by Rolle's Thm). For x in $[a,b]$ we define

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x)-g(a)\big)-\big(f(b)-f(a)\big)$$

Then h is continuous on $[a,b]$,

differentiable on (a, b),

and $h(a) = h(b) = 0$.

Therefore Rolle's Thm.

implies that there is a

point c in (a, b) such that

$$0 = h'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) - f'(c).$$

Since $g'(c) \neq 0$, we can divide

by $g'(c)$ to obtain the desired

result.