6.3 L'Hospital's Rules

There are several rules, all having to with computing $\lim f(x)/g(x)$.

Rule 1: Suppose $-\infty \leq a < b \leq \infty$ let f, g be differentiable on (a, b) and suppose that $g'(x) \neq 0$ for all $x \in (a, b)$

Suppose that

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x).$$

There are two cases:

(a) If $\displaystyle\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$,

then $\displaystyle\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

(b) If $\displaystyle\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \infty$ or $-\infty$

Then $\displaystyle\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$ or $-\infty$

Pf. If $a < \alpha < \beta < b$, then

Rolle's Thm. implies $g(\beta) \neq g(\alpha)$.

Also the Cauchy Mean Value Thm

implies that there exists $u \in (\alpha, \beta)$

such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)} \tag{2}$$

Case (a) If $L \in \mathbb{R}$ and if $\varepsilon > 0$

is given, then there exists $c \in (a, b)$

such that

$$L - \varepsilon < \frac{f'(u)}{g'(u)} < L + \varepsilon, \quad \text{for } u \in (a, c).$$

If we combine this with (2), we obtain

(3) $L - \varepsilon <$

If we take the limit in (3), we obtain

$$L - \varepsilon < \frac{f(\beta)}{g(\beta)} < L + \varepsilon, \quad \text{for } \beta \in (a, c]$$

(Note that if $g(\alpha) = 0$ for some $\alpha \in (a, c]$, then Rolle's Thm. would imply $g'(d) = 0$ for some $d \in (a, c)$ which contradicts our hypothesis). Since $\varepsilon > 0$ is arbitrary, the assertion follows.

Now we turn to Case (a)

If $L = +\infty$ and if M is given,

then there is $c \in (a,b)$

such that

$$\frac{f'(v)}{g'(v)} > M \qquad \text{for } v \in (a,c),$$

Combining this with (2), we obtain

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > M, \qquad \text{for } a < \alpha < \beta < c.$$

If we take the limit

as $\alpha \to a^+$, $\{$as in Case (a), $g(\beta) \neq 0$ for all $\beta \in (a, c)\}$

we have

$$\frac{f(\beta)}{g(\beta)} \geq M \qquad \text{for } \beta \in (a, c).$$

Since M is arbitrary, the assertion follows.

When $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)} = -\infty$

the argument is similar.

Remark. Instead of proving

that $\lim\limits_{x \to a^+} \dfrac{f(x)}{g(x)} = L$,

we can apply virtually the

same argument to show

that $\lim\limits_{x \to b^-} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to b^-} \dfrac{f'(x)}{g'(x)}$

If we want to prove a

2-sided limit such as

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}.$$

we just have to prove

limits on both sides, such

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{f'(x)}{g'(x)}$$

and $\displaystyle \lim_{x \to 0^-} \frac{f(x)}{g(x)} = \lim_{x \to 0^-} \frac{f'(x)}{g'(x)}.$

Similarly, using right-handed limits, we can prove that (under the corresponding hypotheses)

if $\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)} = L$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

and then verify that both one-sided limits have the same value.

Moreover, the hypothesis of Rule 1 allows a^+ to be $-\infty$. This means that if

$$\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = L \text{, then}$$

$$\lim_{x \to -\infty} \frac{f(x)}{g(x)} = L.$$

Ex. Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Ex. Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

$$= \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} -\frac{\cos x}{2}$$

$$= -\frac{1}{2}.$$

We can use the functions e^x and $\ln x$ to prove many (otherwise difficult) limits.

Ex. Compute $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$.

In fact, let $g(x) = \ln\left((1+x)^{\frac{1}{x}}\right)$

$$= \frac{1}{x} \ln(1+x) = \frac{\ln(1+x)}{x}.$$

By applying L'Hopital's

Rule to compute

$$\lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = \frac{1}{1+0} = 1.$$

Since $\lim_{x \to 0} g(x) = 1,$

it follows that

$$\lim e^{g(x)} = e^{\ln(1+x)^{1/x}}$$

$$\lim_{x\to 0} e^{g(x)} = \lim e^{\ln(1+x)^{\frac{1}{x}}}$$

$$= \lim (1+x)^{\frac{1}{x}} = e^1 = e.$$

Ex. Compute $\lim_{x\to\infty} x^{1/x}$, ~~there~~

We apply $\ln x$ as above

Set $g(x) = \ln\left[x^{1/x}\right]$

$= \frac{1}{x}\ln x$. As in the first

example above it follows

Thus, we have shown that

$$\lim_{x \to 0} g(x) = 0$$

Since e^x is a continuous function, we get

$$e^{g(x)} = e^{\ln(x^{1/x})} = x^{\frac{1}{x}}$$

$\downarrow$

approaches $e^0 = 1$

Hence $\lim_{x \to \infty} x^{1/x} = 1$

Section 6.4 Taylor's Theorem.

Suppose that a polynomial $P(x)$ can be written as

$$P(x) = \sum_{n=0}^{N} a_n x^n .$$

How do we write $P(x)$ as $\sum_{n=0}^{N} c_n (x-a)^n$.

How do we write c_n in terms of $P(x)$?

Thus, suppose

$$p(x) = \sum_{n=0}^{N} c_n (x-a)^n$$

$$p(a) = c_0$$

$$p'(x) = \sum_{n=0}^{N} c_n n (x-a)^{n-1}$$