Taylor's Theorem.

Suppose that f has $n+1$ continuous derivatives in $[a, b]$. Then one can write

$\cancel{f(b) = \frac{1}{n!}}$

$$f(b) = f(a) + \frac{f'(a)(x-a)}{1!} + \cdots \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(b),$$

where

$$R_n(b) = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt.$$

To do this, we integrate

by parts:

Set $u = f^{(n)}(t)$, $v' = (b-t)^{n-1}$

and $u' = f^{(n+1)}(t)$, $v = \frac{-(b-t)^n}{n}$.

Hence, $R_{n-1}(t)$ satisfies

$$R_{n-1}(t) = \frac{1}{(n-1)!}\int_a^b \frac{f^{(n)}(t)\,(b-t)^{n-1}}{n}$$

$$= -\frac{1}{n!} f^{(n)}(t)(b-t)^n \Big|_{t=a}^{t=b}$$

$$+ \frac{1}{n!} \int_a^b f^{(n+1)}(t)(b-t)^n \, dt$$

$$= \frac{1}{n!} f^{(n)}(a)(b-a)^n$$

$$+ \frac{1}{n!} \int_a^b f^{(n+1)}(t)(b-t)^n \, dt$$

$$\therefore R_{n-1}(b) = \frac{f^{(n)}(a)(b-a)^n}{n!}$$

$$+ R_n(b)$$

Hence

$$f(b) = f(a) + \frac{f'(a)}{1!} + \dots f^{(n)}(a)(b-a)$$

$$+ \; R_n(b), \quad \text{i.e.,}$$

$$f(b) = P_n(b) + R_n(b).$$

We can use this to show

that $\lim_{n\to\infty} P_n(x) = f(x)$

for various functions.

Ex Let $f(x) = \sin x$, or $\cos x$

Since $|f^{(n+1)}(z)| \leq 1$,

it follows that

$|P_n(x)| \leq \frac{|x|^n}{n!}$ as $n \to \infty$

it follows that $\sin x$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

and also that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

as $n \to \infty$

Darboux Integral.

Given a bounded function $f: I \longrightarrow \mathbb{R}$, we define the lower integral of f on I by

$$L(f) = \sup \left\{ L(f:P) : P \in \mathcal{P}(I) \right\}$$

where $\mathcal{P}(I)$ is the set of partitions of I. Similarly we define the upper integral by

$$U(f) = \inf \left\{ U(f;P) : P \in \mathcal{P}(I) \right\}.$$

Thm. The lower integral $L(f)$ and the upper integral $U(f)$ on I both exist. Moreover $L(f) \le U(f)$. (4)

If P_1 and P_2 are any pair of partitions of I, then

then it follows that

$L(f: P_1) \leq U(f; P_2)$.

$\therefore$ the number $U(f: P_2)$ is

an upper bounded for

the set $\{L(f: P); P \in \mathcal{P}(I)\}$

Hence, $L(f)$, being the

supremum of the set satisfies

$L(f) \leq U(f: P_2)$.

Since P_2 is an arbitrary partition of I, then $L(f)$ is a lower bound for the set $\left\{ U(f:P) : P \in \mathcal{P}(I) \right\}$.

Hence the infimum of this set set satisfies $L(f) \leq U(f)$.

Def'n. Let $f: I \to \mathbb{R}$ be a bounded function I. We say

f is Darboux integrable on I if $L(f) = U(f) = \int_a^b f$

Ex. Remember how hard it was to calculate $\int_0^3 g$ for the function $g(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ 3 & \text{if } 1 < x \le 3 \end{cases}$

For $\varepsilon > 0$, we define

$P_\varepsilon = (0, 1, 1+\varepsilon, 3)$. We get the upper sum

$$U(g:P_\varepsilon) = 2\cdot(1-0) + 3(1+\varepsilon-1)$$

$$+ 3(2-\varepsilon)$$

$$= 2 + 3\varepsilon + 6 - 3\varepsilon = 8.$$

Therefore, $U(g) \le 8$.

(Recall $U(g)$ is the infimum of

all partitions of $[0,3]$.)

Similarly the lower sum is

$$L(g:P_\varepsilon) = 2 + 2\varepsilon + 3(2-\varepsilon) = 8-\varepsilon$$

so that $L(g) \geq 8$. Then

$$8 \leq L(g) \leq U(g) = 8.$$

which means $L(g) = U(g) = 8$

$\therefore$ The Darboux integral of g is $\int_0^3 g = 8$.

Integrability Criterion.

Let $I = [a,b]$ and let

$f: I \to \mathbb{R}$ be a bounded fcn.

on I. Then f is Darboux

integrable if and only if

for each $\varepsilon > 0$, there is a

partition P_ε of I such that

$$U(f; P_\varepsilon) - L(f; P_\varepsilon) < \varepsilon. \quad (5)$$

Pf. If f is integrable, then we have $L(f) = U(f)$. If $\varepsilon > 0$ then since the lower integral is a supremum, there is a partition P_1 of I such that

$$L(f) - \frac{\varepsilon}{2} < L(f; P_1).$$

Similarly there is a partition

P_2 of I such that

$$U(f; P_2) < U(f) + \frac{\varepsilon}{2}.$$

If we let $P_\varepsilon = P_1 \cup P_2$, then

P_ε is a refinement of

P_1 and P_2. Hence

$$L(f) - \frac{\varepsilon}{2} < L(f; P_1) \leq L(f: P_\varepsilon)$$

$$\leq U(f; P_\varepsilon) \leq U(f: P_2) < U(f) + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad U(f:P_\varepsilon) < U(f) + \frac{\varepsilon}{2} \quad \text{and}$$

$$-L(f;P_\varepsilon) < -L(f) + \frac{\varepsilon}{2}$$

Adding together and using $U(f) = L(f)$,

$$U(f:P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon .$$

For the converse, note that

$$L(f:P) \leq L(f) \quad \text{and}$$

$$U(f) \leq U(f;P_\varepsilon) .$$

Hence

$U(f) - L(f) \leq U(f;P) - L(f;P)$

Now for each $\varepsilon > 0$, suppose there is a partition P_ε such that (5) holds. Then we have

$U(f) - L(f) \leq \varepsilon$.

Since ε is arbitrary, we conclude $U(f) \leq L(f)$. But we have $L(f) \leq U(f)$ is always true, so we have

$U(f) - L(f) \leq 0$ and

$U(f) - L(f) \geq 0$.

It follows $U(f) = L(f)$,

so f is Darboux integrable