1.3 Infinite Sets.

A set S is <u>denumerable</u>

if there is a bijection

$f : \mathbb{N} \longrightarrow S$

If we write $x_n = f(n)$,

for all $n = 1, 2, \ldots$ then

$S = \{ x_n : n = 1, 2, 3, \ldots \}$,

where $x_j \neq x_k$ if $j \neq k$.

Ex. Some examples.

The set $E = \{2n : n \in N\}$

of even natural numbers

is denumerable.

So is $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

So is $P: \{2, 3, 5, 7, 11, \ldots\}$

(the set of prime numbers).

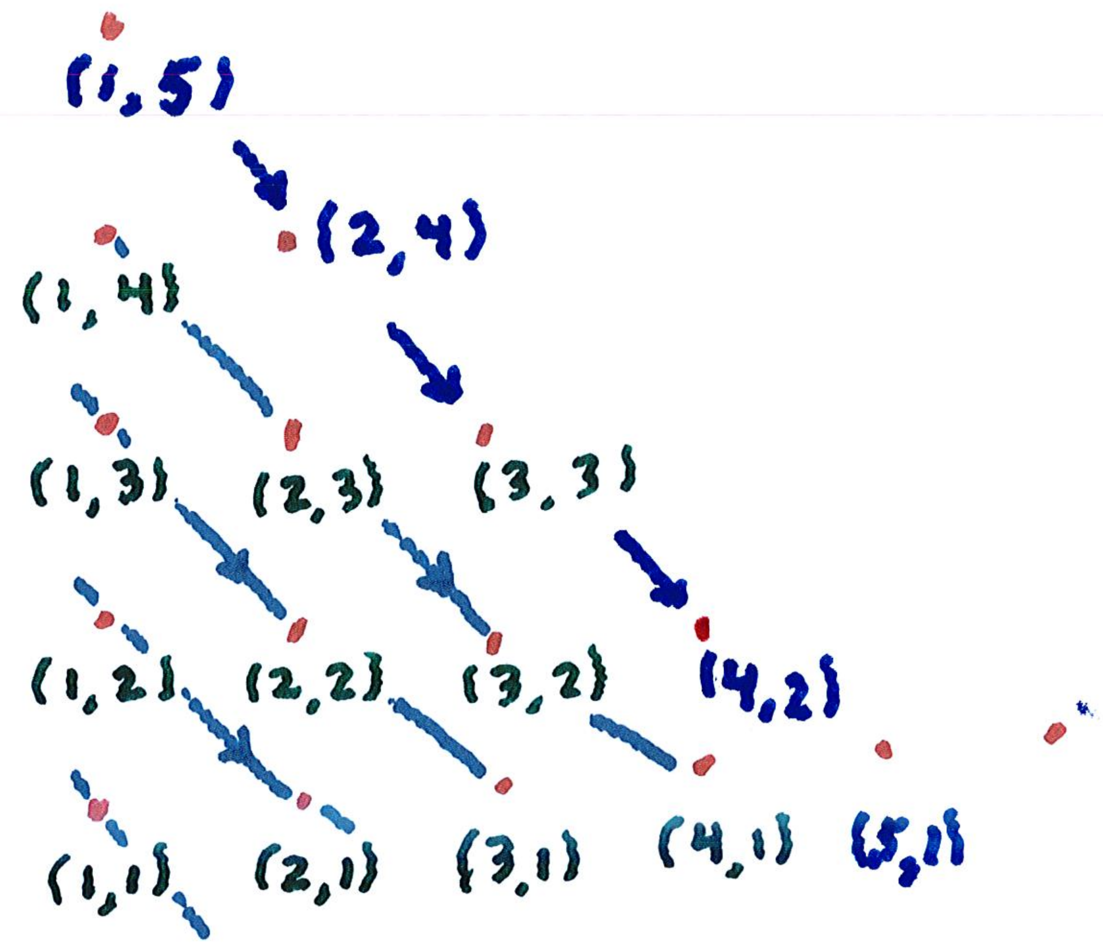
$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

Show $\mathbb{Z}$ is denumerable

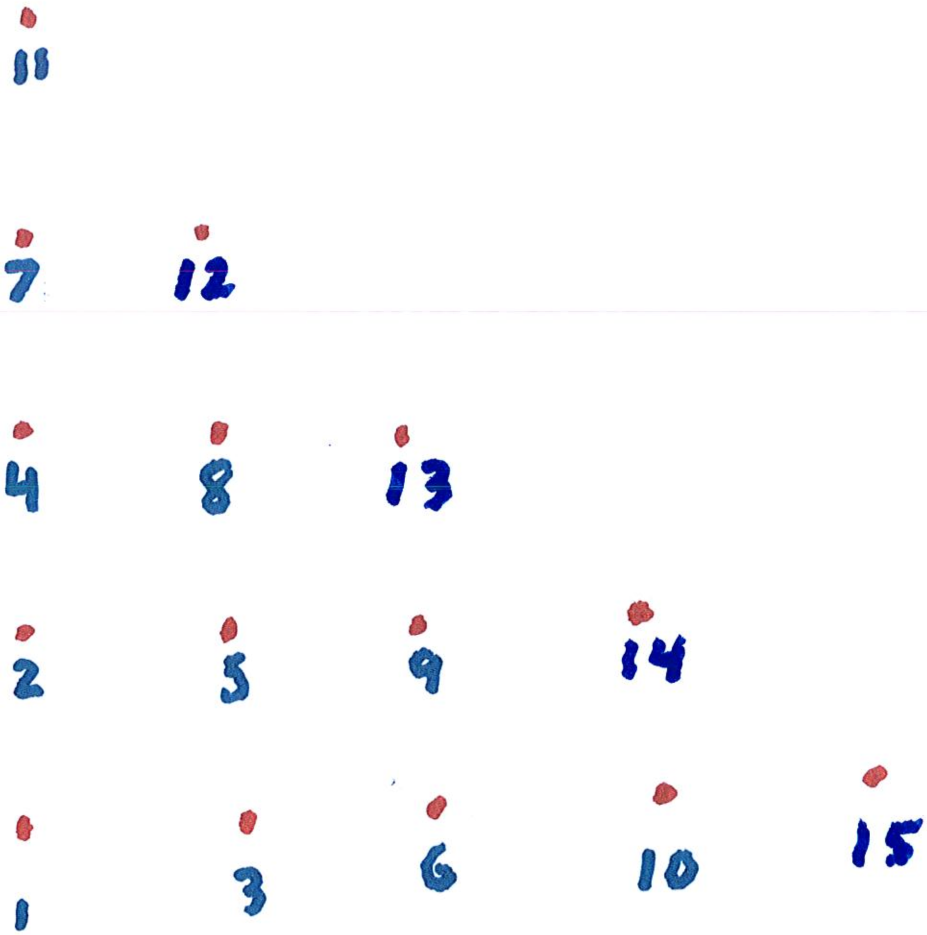
$$\begin{cases} f(n) = \frac{n}{2} & \text{if } n \text{ is even} \\ f(n) = -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

is the formula for the bijection of $\mathbb{N}$ onto $\mathbb{Z}$.

Is $N \times N$ denumerable?



Follow first diagonal, then the second, then the third, etc. .


11
7 12
4 8 13
2 5 9 14
1 3 6 10 15

Using this method, let

$f(m,n)$ = value assigned

to (m,n).

Thus $f(1,1) = 1$ $\quad f(1,2) = 2$

$f(2,1) = 3$. $\quad f(1,3) = 4$

$\ldots f(4,1) = 10, \quad \ldots$

Number of first 2 diagonal terms

$= 1 + 2 = 3 \quad f(2,1) = 3$

Number of k diagonal terms is

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

$$f(k,1) = \frac{k(k+1)}{2}.$$

Observe that as we move along the path, $f(m,n)$ increases by 1 with each step. Therefore,

$$f: N \times N \to N \text{ is 1-to-1 and onto}$$

It follows that f has an inverse $g: N \to N \times N$ that is also 1-to-1 and onto.

g satisfies

$g(1) = (1,1)$

$g(2) = (1,2)$

$g(3) = (2,1)$

$g(4) = (1,3)$, etc.

In general

$g(k) = (m(k), n(k))$

for $k = 1, 2, \ldots$

Now define a

function $\pi(m, n) = \dfrac{m}{n}$

and also define

$$h(k) = \pi(g(k)) = \frac{m(k)}{n(k)}$$

This is the k-th positive

rational number at

the k-th point on

the path.

Thus we obtain a function $h : N \to Q^+$

that is <u>onto</u> but <u>not 1-to-1</u>.

We want to modify h to make it <u>1-to-1 and onto</u>.

$$h(1) = \left(\frac{1}{1}\right) = 1$$

$$h(5) = \left(\frac{2}{2}\right) = 1$$

Idea: We have a path

$h: \mathbb{N} \longrightarrow \mathbb{Q}^+$ that runs

through all rational numbers

We should delete all

rational numbers that

already occur on the

list.

$\frac{1}{5}$ 10	$\frac{2}{5}$	$\frac{3}{5}$		
$\frac{1}{4}$ 6	$\frac{2}{4}$ x	$\frac{3}{4}$	$\frac{4}{4}$	
$\frac{1}{3}$ 4	$\frac{2}{3}$ 7	$\frac{3}{3}$ x	$\frac{4}{3}$	
$\frac{1}{2}$ 2	$\frac{2}{2}$ x	$\frac{3}{2}$ 8	$\frac{4}{2}$ x	
$\frac{1}{1}$ 1	$\frac{2}{1}$ 3	$\frac{3}{1}$ 5	$\frac{4}{1}$ 9	$\frac{5}{1}$ 11

We delete $\frac{m}{n}$

if the rational number

$\frac{m}{n}$ already occurs on the list

Thus, we obtain a function

$H : N \longrightarrow Q^+$ that is 1-to-1

and onto:

$H(1) = \frac{1}{1}$ $\qquad$ $H(7) = \frac{2}{3}$

$H(2) = \frac{1}{2}$ $\qquad$ $H(8) = \frac{3}{2}$

$H(3) = \frac{2}{1}$ $\qquad$ $H(9) = \frac{4}{1}$

$H(4) = \frac{1}{3}$ $\qquad$ $H(10) = \frac{1}{5}$

$H(5) = \frac{3}{1}$ $\qquad$ $H(11) = \frac{5}{1}$

$H(6) = \frac{1}{4}$ $\qquad$ $H(12) = \frac{1}{6}$, etc.

Thus, the function

$H: N \to Q^+$ provides a list

of all positive rational

numbers such that each

rational number occurs exactly

once on the list. Thus,

H is 1-to-1 and onto.

Hence Q^+ is denumerable.

If we write $\mathbb{H}(k) = r_k$.

for $k = 1, 2, \ldots,$ then

$$\mathbb{Q}^+ = \left\{ r_1, r_2, r_3, \ldots \right\}$$

Now we write

$$\begin{array}{lll}
(r_1, r_4) & & \\
(r_1, r_3) & (r_2, r_3) & (r_3, r_3) \\
(r_1, r_2) & (r_2, r_2) & (r_3, r_2) \\
(r_1, r_1) & (r_2, r_1) & (r_3, r_1)
\end{array}$$

This is a list Q_2^+ of all ordered pairs of positive rational numbers. We conclude Q_2^+ is also denumerable. Letting R_k be the k-th element of this list, consider

(R_1, r_4)

(R_1, r_3) $\quad (R_2, r_3)$ $\quad (R_3, r_3)$

(R_1, r_2) $\quad (R_2, r_2)$ $\quad (R_3, r_2)$

(R_1, r_1) $\quad (R_2, r_1)$ $\quad (R_3, r_1)$

This provides a list of all ordered triples of positive rationals.

Hence

Q_3^+ is denumerable.

Sets can be arbitrarily large: For any set S, let

$\mathcal{P}(S)$ be the set of all subsets of S.

Cantor's Thm:

There does NOT exist a map $\varphi : S \longrightarrow \mathcal{P}(S)$ that is onto.

Proof. Suppose

$$\varphi : S \longrightarrow \mathcal{P}(S)$$

is a surjection.

Since $\varphi(x)$ is a subset of S. Either x belongs to $\varphi(x)$ or it does not belong to $\varphi(x)$. We let

$$D = \{ x \in S : x \notin \varphi(x) \}$$

Since ϕ is a surjection,

there exists $x_0 \in S$ such that $\phi(x_0) = D$.

There are 2 cases:

1. Suppose $x_0 \in D$.

   Then $x_0 \in \phi(x_0)$.

   By definition of D,

   $x_0 \notin D$. Contradiction

2. Suppose $x_0 \notin D$.

Then $x_0 \notin \varphi(x_0)$.

By definition of D,

$x_0 \in D$. Contradiction.

Ex. Suppose $S = \{a, b, c\}$

$\wp(S) = \{ \phi, \{a\}, \{b\}, \{c\},$

$\{a,b\}, \{a,c\}, \{b,c\}$

and $\{a,b,c\} \}$

Ex. Use Induction to show that

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Let $P(n)$ be the above statement. When $n=1$, this means

$$1 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1$$

Thus $P(1)$ is true.

Now assume that $P(n)$ is true

Then

$$1^3 + 2^3 + \dots + n^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

by the inductive assumption

$$= \frac{n^2}{4}(n+1)^2 + \frac{4(n+1)(n+1)^2}{4}$$

$$= \frac{(n+1)^2}{4}\left(n^2 + 4(n+1)\right)$$

$$= \frac{(n+1)^2}{4}\left(n^2 + 4n + 4\right)$$

$$= \frac{(n+1)^2 (n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

This proves $P(n+1)$ is true.

Thus $P(n)$ is true for all $n \in \mathbb{N}$.