Today we prove:

Fundamental Thm. of Algebra

Given any positive integer $n \geq 1$, and any complex numbers $a_0, a_1, \ldots, a_n$ such that $a_n \neq 0$, the polynomial equation

$$a_n z^n + \ldots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

We use the Extreme Value Theorem for real-valued functions of two real variables.

Thm. (Extreme Value Theorem)

Let $f : D \to \mathbb{R}$ be a continuous function on the closest disk

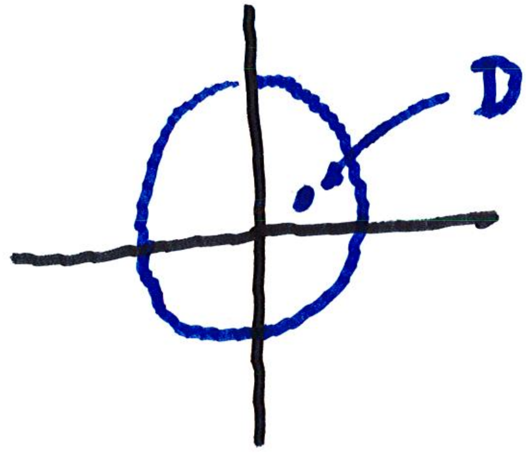
$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq R^2.$$

Then f is bounded and attains its minimum and maximum values on D. In other words, there exist points

$x_m, x_M \in D$ such that

$$f(x_m) \leq f(x) \leq f(x_M)$$

for all x in D.



If we define a polynomial

$f : \mathbb{C} \to \mathbb{C}$ by setting

$$f(z) = a_n z^n + \cdots + a_1 z + a_0 ,$$

then note that we can regard

$$(x, y) \to |f(x+iy)| \text{ as a function}$$

from $\mathbb{R}^2$ to $\mathbb{R}$.

We may also denote this function by $|f(\cdot)|$ or $|f|$.

It is a composition of continuous functions (polynomials and the square root), and therefore it is also continuous.

Lemma. Let $f: \mathbb{C} \to \mathbb{C}$ be any polynomial. Then there is a point z_0 in $\mathbb{C}$ where the function $|f|$ attains its minimum value in $\mathbb{R}$.

Proof of Lemma. If f is a constant polynomial polynomial function, then the statement of the lemma is true since $|f|$ attains its minimum at every point in $\mathbb{C}$. So, choose $z_0 = 0$.

If f is not constant, then

the degree of the polynomial is at least 1. In this case, we set

$$f(z) = a_n z^n + \dots + a_1 z + a_0,$$

with $a_0 \neq 0$. Now, assume $z \neq 0$, and set

$$M = \max\left\{ |a_0|, \dots, |a_n| \right\}$$

We can obtain a lower bound for $|f(z)|$ as follows:

$|f(z)|$

$$= |a_n||z|^n \left| 1 + \frac{a_{n-1}}{a_n} \cdot \frac{1}{z} + \dots + \frac{a_0}{a_n} \frac{1}{z^n} \right| .$$

Since $\lim_{z \to \infty} \frac{1}{z} = 0$, it follows

that for large $|z|$, we have

$$\left| \frac{a_{n-1}}{a_n} \cdot \frac{1}{z} + \dots + \frac{a_0}{a_n} \cdot \frac{1}{z^n} \right| \le \frac{1}{2} .$$

Thus, if $|z| \geq R$ for large

R, we have

$$|f(z)| \geq \frac{|a_n||z|^n}{2}, \quad \text{if } |z| \geq R.$$

By choosing $|z| \geq R$, for large R, it follows that

$$|f(z)| \geq \frac{|a_n| R^n}{2} > |f(0)|.$$

Let $D \subset \mathbb{R}^2$ be the disk of radius R about 0, and define a function $g: D \to \mathbb{R}$ by $g(x,y) = |f(x+iy)|$.

Then g is continuous, so

we can apply the Exteme Value Theorem in order to obtain a point $(x_0, y_0) \in D$ such that g attains its minimum at (x_0, y_0)

By the choice of R, we have that for $z \in \mathbb{C} \setminus D$,

$$|f(z)| \geq g(0,0) \geq g(x_0, y_0)$$

such that g attain its minimum at (x_0, y_0). By the choice of R we have that for $z \in \mathbb{C} \setminus D$,

$$|f(z)| > |g(0,0)| \geq |g(x_0, y_0)|$$

Therefore $|f|$ attains its minimum in $z = x_0 + iy_0$.

This proves the lemma.

Proof of theorem.

Let $z_0 \in \mathbb{C}$ be a point

where the minimum is attained.

There are 2 cases:

Case I. $f(z_0) \neq 0$. and

Case II. $f(z_0) = 0$.

In Case I, we have that

$$|f(z_0)| \leq |f(z)|, \quad z \in \mathbb{C}$$

We define a new function

$g : \mathbb{C} \to \mathbb{R}$ by $\quad g(z) = \dfrac{f(z+z_0)}{f(z_0)}$.

Note that g is a polynomial of degree n and the minimum of $|f|$ is attained at $z=0$.

In fact,

$$|g(z)| = \frac{|f(z+z_0)|}{|f(z_0)|} \geq \frac{|f(0+z_0)|}{|f(z_0)|} = g(0).$$

Note also that $g(0) = 1$.

It follows that

$$g(z) = b_n z^n + \cdots + b_k z^k + 1.$$

with $n \geq 1$ and

$b_k \neq 0$, for some k,

with $1 \leq k \leq n$.

Let $b_k = |b_k| e^{i\theta}$, and

consider z of the form

$$z = r |b_k|^{-\frac{1}{k}} e^{i(\pi - \theta)/k}$$

with $r > 0$.

Note that if we take

k-th powers,

$$z^k = r^k |b_k|^{-1} \cdot e^{i(\pi - \theta)}$$

OR:

$$|b_k| z^k = -e^{-i\theta} r^k .$$

OR:

$$b_k z^k = -r^k .$$

For z of this form, we have

$$g(z) = 1 - r^k + r^{k+1} h(r)$$

where h is a polynomial.

Then, for $r < 1$,

the Triangle Property implies

$$|g(z)| \leq 1 - r^k + r^{k+1} |h(r)|.$$

Since $|r^{k+1} h(r)| \le C r^{k+1}$

$\le C r \cdot r^k < \frac{1}{2} r^k$ for

small r, we conclude that

$$g(z) < 1 - r^k + \frac{1}{2} r^k$$

$$= 1 - \frac{r^k}{2}. \quad \text{for small } r$$

Thus $g(z) < 1$, which

contradicts the assumption

that $g(z) \ge g(0) = 1$.

Thus Case I is not possible.

The remaing property is Case II. This implies that $f(z_0) = 0$, which means f has a root z_0.