2.1 Algebraic and Order Properties of $\mathbb{R}$.

On $\mathbb{R}$, there are two operations; addition $+$ multiplication $\cdot$. They satisfy:

(A1) $a+b=b+a$, (commutative addition)

(A2) $(a+b)+c=a+(b+c)$

(associative addition)

(A3) There is an element 0 in $\mathbb{R}$ so $a+0=a$

(0-element exists)

(A4) For each a in $\mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ so that

$a+(-a)=0$ and $(-a)+a=0$

(negative element)

(M1) $a\cdot b = b\cdot a$ (commutative multiplication)

(M2) $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

(associative multiplication)

(M3) There is an element 1 in $\mathbb{R}$

so that $a \cdot 1 = 1 \cdot a = a$

(unit element exists)

(M4). For each $a \neq 0$ in $\mathbb{R}$,

there exists an element

$1/a$ such that

$a \cdot \left(\frac{1}{a}\right) = 1$ and

$(1/a) \cdot a = 1$

(existence of reciprocal)

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

and

$(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

(distributive property)

In a word, $\mathbb{R}$ is a field

By applying some of the above properties, one can show that the

(1) zero element 0, the

(2) unit element 1, and

(3) the reciprocal $\frac{1}{a}$ are

all unique.

For example, suppose $a \neq 0$

and $a \cdot b = 1$. Then

$$\underset{(M_3)}{b =} 1 \cdot b \underset{(M4)}{=} \left(\left(\frac{1}{a}\right) \cdot a \right) \cdot b$$

$$\underset{(M2)}{=} \left(\frac{1}{a}\right) \cdot (a \cdot b) \underset{(Hyp.)}{=} \left(\frac{1}{a}\right) \cdot 1 \underset{(M3)}{=} \frac{1}{a}$$

This proves (3)

Also, if $a \in \mathbb{R}$, then $a \cdot 0 = 0$

In fact,

$$a + a \cdot 0 \underset{\text{by } (M_3)}{=} a \cdot 1 + a \cdot 0 \underset{\text{by } (D)}{=} a \cdot (1 + 0)$$

$$\underset{\text{by } (A_3)}{=} a \cdot 1 \underset{\text{by } (M_3)}{=} a$$

Adding $(-a)$ to both sides, we get

$a \cdot 0 = 0.$

Also, $0 = (-1)(-1+1) = (-1)(-1) + (-1)$.

Adding 1 to both sides, we get

$(-1)(-1) = 1$

We define subtraction by

$$a-b = a+(-b)$$

and also we write

$$ab = a\cdot b,$$

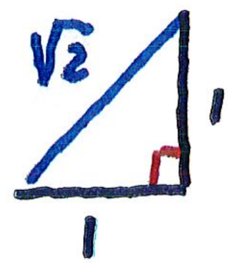
and $a^2 = aa$ and

$a^3 = a^2 a$ and

$a^{n+1} = a^n a$, etc.

$\mathbb{Q}$, $\mathbb{R}$ are both fields.

Thm. There does not exist a rational number r such that $r^2 = 2$



Suppose by contradiction that $r = p/q$. Then

$$r^2 = (p/q)^2 = 2 \rightarrow p^2 = 2q^2.$$

We can assume that

p and q have no common

factor. Then at most one of p and q is even.

Since $p^2 = 2q^2$, we see that p^2 is even. This implies that p is also even (because if $p = 2n+1$ is odd, then $p^2 = 4n^2 + 4n + 1$ is also odd.)

Hence we can write $p = 2m$,

so that

$$p^2 = 4m^2 = 2q^2.$$

Dividing by 2,

$$2m^2 = q^2.$$

Hence q^2 must be even,

which implies q is even.

This shows that both

p and q are even, which

is a contradiction.

It follows that

$\mathbb{R}$ must include numbers

that are irrational

(i.e., not rational).

For this purpose we need to

study Order Properties.

i.e., $<$ and $>$.

Order Properties of $\mathbb{R}$

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers such that

(i) If $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$

(iii) If $a \in \mathbb{P}$, then exactly one of the following holds:

$a \in \mathbb{P}$, $a = 0$, $(-a) \in \mathbb{P}$

Trichotomy Property

If $-a \in \mathbb{P}$, we say a is negative, and we write $a < 0$ or $0 > a$.

If $a \in \mathbb{P}$, we write $a > 0$ or $0 < a$

If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$.

If $-a \in \mathbb{P} \cup \{0\}$, then we write $a \leq 0$.

If (i)–(iii) hold, then we say $\mathbb{R}$ is an ordered field.

Applying the Trichotomy Property

to $a-b$, we get

If $a-b \in \mathbb{P}$, i.e. $a > b$.

If $-(a-b) \in \mathbb{P}$, then $(b-a) \in \mathbb{P}$

$\Rightarrow b > a$

If $a-b = 0$, then $a = b$

Here are the Rules for

Inequalities :

Thm. Let $a, b, c \in \mathbb{R}$.
2.1.7

(a) If $a > b$ and $b > c$, then
$a > c$

(b) If $a > b$, then $a + c > b + c$

(c) If $a > b$ and $c > 0$, then
$ca > cb$

If $a > b$ and $c < 0$, then
$ac < cb$

Proof of (a): $a - b > 0$, $b - c > 0$
then $(a - b) + (b - c) > 0$
or $a - c > 0 \rightarrow a > c$

(b) If $a-b>0$, then

$$(a+c)-(b+c) = a-b > 0$$

$$\rightarrow a+c > b+c$$

(c) If $a>b$ and $c>0$, then

$$ca-cb = c(a-b) > 0.$$

$$\rightarrow ca > cb$$

If $c<0$, then $-c>0$. Hence

$$c(b-a) = -c(a-b) > 0$$

$$\rightarrow cb-ca>0 \rightarrow cb > ca.$$

The Order Properties

in 2.1.5 and 2.1.6 lead to 2.1.10 and 2.1.11, which are useful for solving inequalities:

1. Suppose that $ab > 0$. If $a > 0$, then $b > 0$.
2. If $ab > 0$ and $a < 0$, then $b < 0$
3. If $ab < 0$ and $a > 0$, then $b < 0$
4. If $ab < 0$ and $a < 0$, then $b > 0$

We need to prove several facts:

Thm 2.1.8

(a) if $a \in \mathbb{R}$ and $a \neq 0$, then

$a^2 > 0$

(b) if $n \in N$, then $n > 0$

Since $1 = 1^2$, (a) $\Rightarrow$ $1 > 0$

(c) If $n \in N$, then $n > 0$.

Apply (b) and (i) from Order Properties. Use Math. Ind.

Proof of (a). If $a \neq 0$, then either $a > 0$ or $a < 0$.

If $a > 0$, then $a^2 > 0$ (i.e $a \in \mathbb{P}$)

If $a < 0$, then $-a > 0$.

Hence $a^2 = (-a)(-a) > 0$,

since $(-1)(-1) > 0$.

Proof of (b). Since $1 = 1^2$, it follows from (a) that

$1^2 > 0$.

Pf. of (c). If $n \in \mathbb{N}$, then $n > 0$. Clearly $1 > 0$.

Assuming by induction that $n > 0$, then $n+1 > 0$.

It is also important that if $a > 0$, then $a^{-1} > 0$. To see this, suppose that $a^{-1} < 0$. Then $1 = a \cdot a^{-1} < a \cdot 0 = 0$.

Ex. Find all real numbers x such that $3x+4 \le 12$.

Justify each step.

$$3x+4 \le 12 \Leftrightarrow 3x \le 8 \Leftrightarrow x \le \frac{8}{3}$$

By (b) of 2.1.7

By (c) of 2.1.7

Ex. Solve $x^2-4x-5 < 0$.

$x^2-4x-5 = (x-5)(x+1) < 0$

$\Leftrightarrow$ If $x-5 > 0$, then $x+1 < 0$

By Property (3) above

No solution.

Or, by Property 4, if

$x-5<0$, then $x+1>0$.

$\therefore$ Solution is $-1<x<5$.

Finally, we have

Thm. 2.1.8:

(a) if $a \in \mathbb{R}$ and $a \neq 0$,

then $a^2>0$.

(b) $1>0$. Since $1=1^2$

this follows from (a)