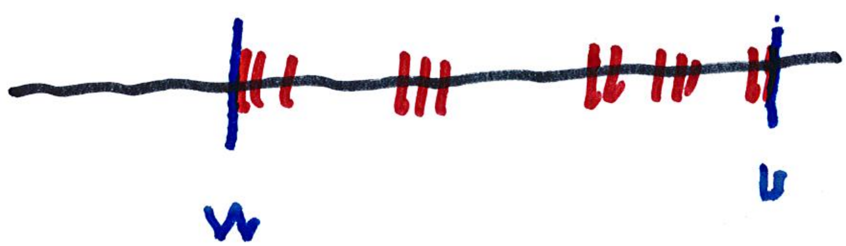
2.3 The Completeness Property

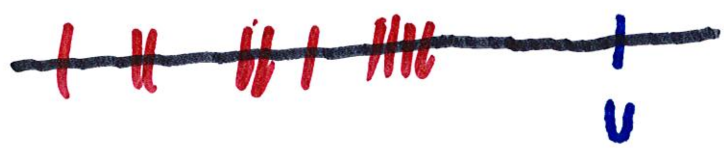
In this section we show that a bounded subset S of $\mathbb{R}$ has a "maximum" u and a "minimum" w.



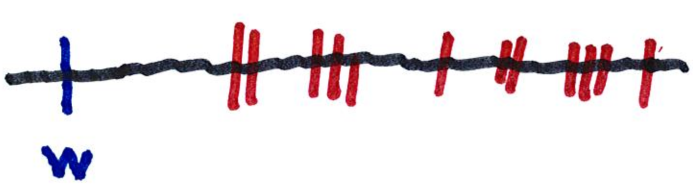
We say that S is bounded above if there is a number u

such that $s \leq u$ for all $s \in S$.

Each such number u is called an <u>upper</u> <u>bound</u> of S



Similarly, we say S is bounded below if there is a number w such that $w \leq s$ for all $s \in S$.



Each such number w is called a lower bound of S.

Example. $S = \{x \in \mathbb{R} ; x < 2\}$ is bounded above but not bounded below.

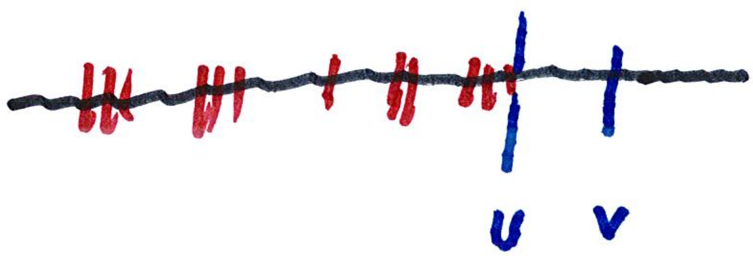
Definition. The number u is a supremum of S (also written as $\sup S$ or least upper bound of S)

if

(1′) u is an upper bound of S and

(2′) if v is any upper bound of S

then $v \geq u$



Similarly, w is an infimum of S

if

(1′) w is a lower bound of S

and

(2′) if t is any lower bound of S,

then $t \le w$

Thus $u = \sup S$, and

$w = \inf S$.

One can show there can only be one supremum of S and one infimum of S.

Suppose there are 2 numbers u_1 and u_2 that are both suprema of S. The fact that

$u_2 = \sup S$ and u_1 is an upper bound of S implies that $u_1 \geq u_2$. The same reasoning implies that $u_2 \geq u_1$.

It follows that $u_1 = u_2$.

Given that u is an upper bound of S, we can express the fact $u = \sup S$ in another way, that is equivalent.

Thm. Let u be an upper bound of S. Then the following statements are equivalent:

(1) If v is an upper bound of S, then $v \geq u$.

(2) If $z < u$, then there is an $s = s_z \in S$, such that $s_z > z$.

We first show that (1) $\Rightarrow$ (2).

Suppose that (2) does not hold. Thus it must be that

$s \le z$ for all $s \in S$.

This implies that z is an upper bound of S, which according to (1), implies that $z \ge u$, which contradicts the assumption that $z < u$. This proves (2).

At this point, we need:

Lemma. Suppose x is a number that satisfies

$$0 \le x < \varepsilon, \qquad \text{for all } \varepsilon > 0. \quad \text{(i)}$$

Then $x = 0$.

Pf. We show that $x > 0$ leads to a contradiction. Thus, we set $\varepsilon = \frac{x}{2}$. Then we get from (i) that $x < \frac{x}{2}$, which is clearly impossible.

Now we prove that (2) $\Rightarrow$ (1).

Let $\varepsilon > 0$. Since $u - \varepsilon < u$,

(2) implies that there is

an $s_\varepsilon \in S$ such that

$s_\varepsilon > u - \varepsilon$. Now let

v be any upper bound of S.

Then $v \geq s_\varepsilon$. If we combine

these inequalities, we

obtain

$$u - \varepsilon < s_\varepsilon \le v ,$$

or $u - v < \varepsilon$, for all ε.

The Lemma implies that

$u - v \le 0$, or $v \ge u$.

This proves (1),

which proves the theorem

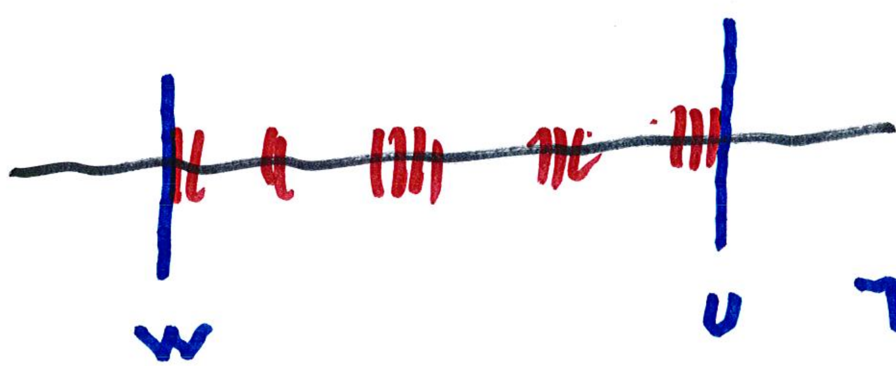
One can show from the construction of $\mathbb{R}$, that the following is true:

Completeness Property of $\mathbb{R}$.

(α) If S is any subset of $\mathbb{R}$ that S is bounded above, then there is a number u such that $u = \sup S$.

Similarly

(B) If S is any subset of $\mathbb{R}$ that is bounded below then there is a number w such that $w = \inf S$



This set S is bounded.

Example. Let $S = [a, b)$.

i.e. $a \leq s < b$. (1)

We first show that $\sup S = b$.

Since $s < b$, it follows that

b = an upper bound of S.

Let $v \in [a, b)$. Set $s = \frac{v+b}{2}$.

This implies $v < s$. Therefore

$v \neq$ an upper bound of S.

Now let $v < a$. If we set $s = a$. Then $v < s$. Then v is not an upper bound of S.

Thus, if $v < b$, then v is NOT an upper bound of S

Hence, if v is an upper bound, then $v \geq b$. It fallows that $\sup S = b$.

Now we show that $\inf S = a$.

Note that (1) implies that

a is a lower bound of S

Now suppose that t is any lower bound of S. Then

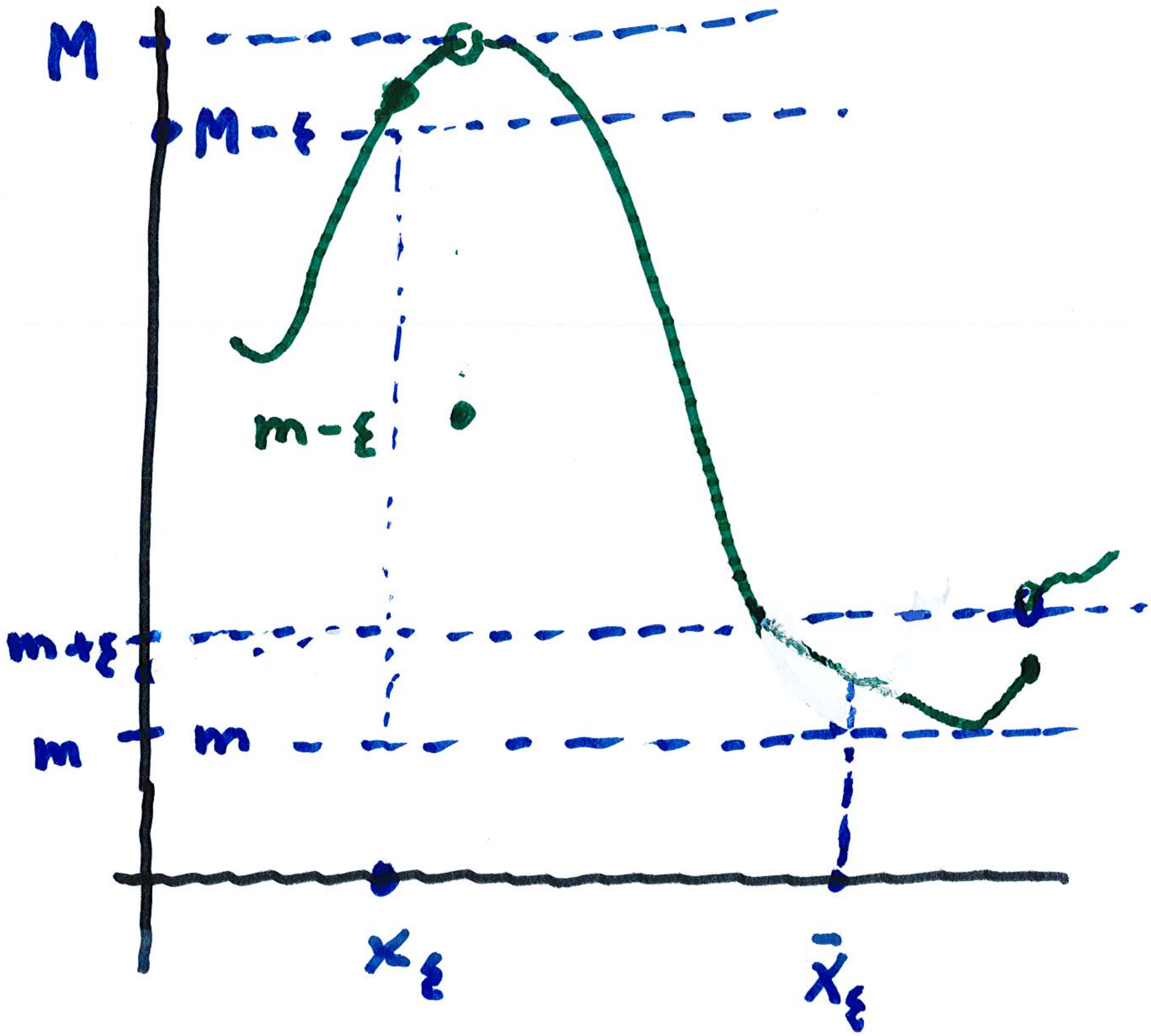
$t \le s$, for all $s \in S$.

In particular, if we set $s = a$, we get $t \le a$. Hence $\inf S = a$

Ex. Let f be a function on an interval I such that there is a constant A such that $|f(x)| \le A$, for all $x \in I$.

Note that f is bounded above by A and bounded below by $-A$. Set $S = \{ f(x) : x \in I \}$

Set $M = \sup S$ and $m = \inf S$



By definition, M is an upper bound, so $f(x) \leq M$, for $x \in I$

Also m is a lower bound, so

$f(x) \geq m$, for all $x \in I$.

For any $\varepsilon > 0$, there is a point $\bar{x}_\varepsilon \in I$, so that

$f(\bar{x}_\varepsilon) < m + \varepsilon$.

Similarly, there is a point x_ε so that $f(x_\varepsilon) > M - \varepsilon$