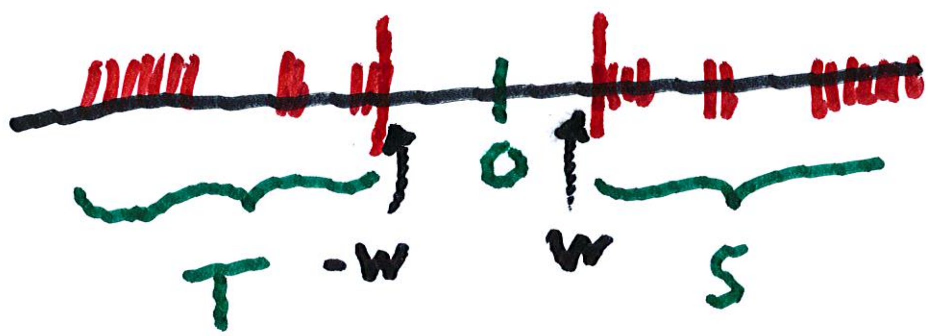
Ex. Let S be a subset of $\mathbb{R}$ that is bounded below.

Let $T = \{-x : x \in S\}$



Then $\inf S = \sup\{-x : x \in S\}$

Pf: Let $w = \inf S$. Then by Criterion 4 on p. 38, a number w is an infimum of S

if (i) w is a lower bound,

and if (ii) for every $\varepsilon > 0$,

there is a $y_\varepsilon \in S$ such that

$$y_\varepsilon < w + \varepsilon.$$

Since w is a lower bound

of S, it satisfies $w \le x$,

for all $x \in S$.

Multiplying by (-1), we get

$$-w \geq -x, \text{ for all } x \in S.$$

Since every element of T is given by $-x$, for $x \in S$, we conclude that $-w$ is an upper bound of T.

Let $\varepsilon > 0$, then

$-y_\varepsilon > -w - \varepsilon$. Again

(Note that $-y_\varepsilon \in T$)

Criterion 4 on p. 38

implies that $-w = \sup T$

$$= \sup \{-x : x \in S\}$$

We conclude that $-w =$

$$\inf S = w = -(-w) = -\sup T$$

$$= -\sup \{-x : x \in S\}$$

This gives the desired equality.

2.5 Intervals

We need to prove a theorem about "nested intervals" before we study 3.4.

We say a sequence of closed bounded intervals are **nested** if

$$I_1 \supseteq I_2 \supseteq \dots \supset I_n \supset I_{n+1} \supset \dots$$

If $I_n = [a_n, b_n]$, then

(b_n) is decreasing, and

(a_n) is increasing, i.e.

we have the picture

We prove the

Nested Interval Property:

Given a sequence of nested closed intervals as above, there is a point η in I_n for all $n \in \mathbb{N}$

Proof. Since $I_n \subseteq I_1$,

we get

$a_n \leq b_n \leq b_1$, for all $n \in \mathbb{N}$.

Hence the sequence (a_n)

is increasing and bounded.

By the Monotone Convergence

Thm., there is an η

satisfying $\eta = \lim(a_n)$.

Clearly $a_n \le \eta$, all $n \in \mathbb{N}$. (1)

We want to show that

$\eta \le b_n$ for all n.

We do this by showing that

for any particular n,

$b_n \ge a_k$, $k = 1, 2, \ldots$

There are 2 cases.

(i) If $n \le k$, then since

$I_n \supseteq I_k$, we have

$a_k \le b_k \le b_n$.

(ii) If $k < n$, then since

$I_k \supseteq I_n$, we have

$a_k \le a_n \le b_n$

We conclude that $a_k \le b_n$ for all k,

so that b_n is an upper bound for

$\{a_k ; k \in N\}$

It follows that

$b_n \geq \eta$, for all $n \in N$,

which implies that

$a_n \leq \eta \leq b_n$, for all $n \in N$,

which in turn implies that

$\eta \in I_n$ for all $n \in N$.

We can use nested intervals to show that the set $\mathbb{R}$ of real numbers is NOT countable.

Suppose that there is a sequence $I = \{x_1, x_2, \dots\}$ such that for any x in $[0,1]$, there is an integer n such that $x_n = x$.

Choose a closed subinterval

$I_1 \subset [0,1]$ such that $x_1 \notin I_1$.

Now choose a closed subinterval

$I_2 \subset I_1$ such that $x_2 \notin I_2$.

In this way we obtain

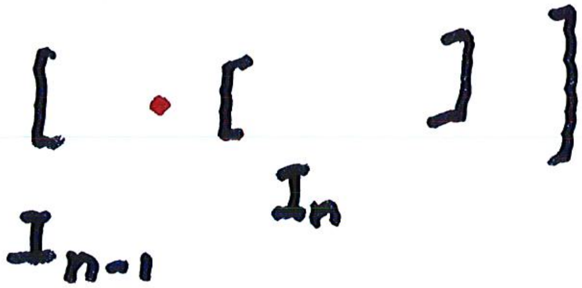
a sequence of closed subintervals

such that

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n$$

such that for all $n = 1, 2, \ldots,$

$x_n \notin I_n$



The Nested Interval Theorem implies that there is a point $\eta \in I_n$, for all $n = 1, 2, \ldots$

Since $x_n \notin I_n$ for all n, it follows that for all $n = 1, 2, \ldots,$

$x_n \neq \eta$.

It follows that $I = [0, 1]$

is <u>not</u> countable.