MA341 Spring 2018 **Theorems for Final Exam**

You must be able to state and prove the
Fundamental Theorem of Calculus.

Theorem. Let f be integrable on $[a, b]$ and let$f$ be continuous at a point $c \in [a, b]$. Then the indefinite integral, defined by

$$F(z) = \int_a^z f \text{ for } z \in [a, b]$$

is differentiable at c and $F'(c) = f(c)$.

Proof. We will suppose that $c \in [a, b]$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\epsilon > 0$ there exists $\eta_\epsilon > 0$ such that if $c \le x \le c + \eta_\epsilon$ then

$$(1)\ f(c) - \epsilon < f(x) < f(c) + \epsilon$$

Let h satisfy $0 < h < \eta_\epsilon$. The Additivity Theorem implies that f is integrable on the intervals $[a, c]$, $[a, c+h]$ and $[c, c+h]$ and that

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Now on the interval $[c, c+h]$ the function f satisfies inequality (1), so that we have

$$(f(c) - \epsilon) \cdot h \le F(c+h) - F(c) = \int_c^{c+h} f \le (f(c) + \epsilon) \cdot h$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \le \epsilon$$

But, since $\epsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{x \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c).$$

It is proved in the same way that the left-hand limit of this difference quotient also equale $f(c)$ when $c \in [a, b]$. QED

I will state the following theorem and you must be able to prove it.

Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is integrable.

Proof. It follows from the Uniform Continuity Theorem that f is uniformly continuous on $[a, b]$. Therefore given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta_\epsilon$ then we have

$$|f(u) - f(v)| < \tfrac{\epsilon}{(b-a)} \ .$$

Let $\mathbb{P} = \{I_i\}_{i=1}^n$ be a partition such that $||\mathbb{P}|| < \delta_\epsilon$. Applying the Maximum-Minimum Theorem we let $u_i \in I_i$ be a point where f attains its minimum value on I_i and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.

Let α_ϵ be the step function defined by $\alpha_\epsilon(x) = f(u_i)$ for $x \in [x_{i-1}, x_i)$ for $i = 1, \dots n-1$ and $\alpha_\epsilon(x) = f(u_n)$ for $x \in [x_{n-1}, x_n]$. Let ω_ϵ be defined similarly using the points v_i instead of the u_i. Then one has

$$\alpha_\epsilon(x) \leq f(x) \leq \omega_\epsilon(x) \text{ for all } x \in [a, b].$$

Moreover, it is clear that

$$0 \leq \int_a^b (\omega_\epsilon - \alpha_\epsilon)$$

$$= \textstyle\sum_{i=1}^n (f(v_i) - f(u_i))(x_{i-1} - x_i)$$

$$= \textstyle\sum_{i=1}^n (\tfrac{\epsilon}{(b-a)})(x_{i-1} - x_i) = \epsilon.$$

Therefore, it follows from the Squeeze Theorem that f is integrable. QED

Math 341 Fall 2017 Study Guide for Final Exam

1. (a) Given a set S of real numbers, define $\sup S = u$.

   (b) Show that for any $\epsilon > 0$ there is a number $x_\epsilon \in S$ such that $u - \epsilon < x_\epsilon \leq u$. pages 37,38

2. Define the Nested Interval Property. page 48

3. Suppose that (x_n) converges to x and (y_n) converges to y. Show that $(x_n y_n)$ converges to xy. pages 61,62

4. Show that if (x_n) is an increasing sequence and that $x_n \leq M$ for all n, then there is a number $L \leq M$ such that $\lim_{n\to\infty} x_n = L$. pages 71, 72

5. Suppose $\Sigma_{n=1}^{\infty} x_n$ is a series with $x_n \geq 0$ such that $\Sigma_{n=1}^{N} x_n \leq M$ for all $N = 1, 2, \ldots$ Show there is an $L \leq M$ such that $\Sigma_{n=1}^{\infty} x_n$ converges to L. page 98

6. Define the Thomae function by setting

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{when } x \text{ is irrational} \end{cases}$$

   Show that f is continuous at x if x is irrational and f is discontinuous at x if x is rational. pages 127, 128

7. Suppose that f is an increasing bounded function on (a, b). Show there is an L such that $\lim_{x\to b^-} f(x) = L$. pages 117,118

8. Show that $S(x) = \sqrt{x}$ is Lipschitz on the interval $[a, \infty)$, where $a > 0$. pages 143, 144 (Hint: Mean Value Theorem)

9. Show that if f is Lipschitz on any interval, then f is uniformly continuous. page 143

10. Find all functions that satisfy $|f(x) - f(y)| \leq |x - y|^2$ when x and y are in an interval I. page 162

11. Suppose that f is differentiable at x_0 and that $f(x_0) \neq 0$. Calculate $(\frac{1}{f})'$ at x_0. page 164

12. Let n be a positive integer and $b > 1$. Use L'Hopital's Rule to show that $\lim_{x\to\infty}(\frac{x^n}{b^x}) \leq M$, i.e., $x^n \leq b^x$, if x is sufficiently large. page 187

13. Use $\ln x$ and L'Hopital's Rule to show that $\lim_{n\to\infty}(1+\frac{a}{n})^{bn} = e^{ab}$. (Hint: set $x = \frac{1}{n}$). page 183

14. If the Taylor polynomial of order 3 of $\ln(1+x)$ is used to calculate the approximate value of $\ln\frac{3}{2}$, what is the maximum error allowed by the Remainder estimate? page 189

15. Show that on every interval $[-d, d]$, the Taylor series of $\cos x$ converges to $\cos x$. page 189

16. Define

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is g continuous at 0? Is g differentiable at 0? Is g uniformly continuous on the interval $[0, 1]$?

17. Suppose f is differentiable at all $x \in (a, b)$. If $f'(x) > 0$ in (a, b) show f is increasing. page 174

18. Let f be a bounded function on $[a, b]$ and let P be a partition of $[a, b]$. What are $U(f, P)$ and $L(f, P)$? What are $U(f)$ and $L(f)$? pages 200, 201

19. State the Integrability Criterion for a function to be Darboux-integrable.

20. Let

$$g(x) = \begin{cases} 3 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } 2 \leq x \leq 4 \end{cases}$$

Use the criterion above with a partition P having just 4 points to show g is Darboux-integrable. pages 228, 229

21. To solve the differential equation $y'(x) = f(x, y(x))$ with $y(x_0) = y_0$, we defined a sequence of curves $y_n(x)$. How are the curves $y_n(x)$ defined? (class notes online)

22. Complete the definition:

    A sequence (x_n) is Cauchy if .... pages 86,87

23. Prove that if (x_n) is Cauchy, then there is an x^* so that $\lim(x_n) = x^*$.

24. Suppose that c is a cluster point of A. Suppose that $\lim_{x\to c} f(x) = L$. Show that if (x_n) is any sequence in A such that $\lim_{n\to\infty}(x_n) = c$ then $\lim_{n\to\infty} f(x_n) = L$. pages 107,108