Answers for Exam 1,
Spring 2017

1. Since (x_n) is bounded, there is a constant $M > 0$ so that

$x_n \le M$ for all $n \in \mathbb{N}$.

Then there is an $\tilde{x} = \sup\{(x_n)\}$

Hence, if $\varepsilon > 0$, there is ~~an~~ a $K \in \mathbb{N}$ so that ~~there is~~ $\tilde{x} - \varepsilon < x_K \le \tilde{x}$.

Since (x_n) is increasing,

$$\tilde{x} - \varepsilon < x_K \le x_n \le \tilde{x} < \tilde{x} + \varepsilon$$

for all $n \ge K$. Hence,

$$\tilde{x} - \varepsilon < x_n < \tilde{x} + \varepsilon,$$

which implies that $\lim_{n \to \infty} x_n = \tilde{x}$.

2. Let $\varepsilon = \frac{|y|}{2}$. Then there is a $K \in \mathbb{N}$, so that if $n \geq K$, then $|y_n - y| < \frac{|y|}{2}$. The "backward Triangle Property implies that

$$|y_n| = |(y_n - y) + y|$$
$$\geq |y| - |y_n - y| > |y| - \frac{|y|}{2}$$
$$= \frac{|y|}{2}.$$

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3. $$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right|$$

$$= \frac{1}{|y_n|} \cdot \frac{|y_n - y|}{|y|} < \frac{2}{|y|} \frac{|y_n - y|}{|y|},$$

where in the ~~second~~ inequality

we have used the fact from #2 that $\frac{1}{|y_n|} < \frac{2}{|y|}$, which holds

when $n \geq K$.

If $\varepsilon > 0$, there is a K_1 so that if $n \geq K_1$, then

$$|y_n - y| < \frac{|y|^2}{2}\varepsilon.$$

Thus if $K_2 = \text{Max}\{K, K_1\}$, then if $n \geq K_2$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \frac{2}{|y|^2} \cdot \frac{|y|^2}{2}\varepsilon = \varepsilon$$

4. Since $\lim y_n = +\infty$, for any $\alpha > 0$, there is a $K \in N$, so that if $n \geq K$, then

$y_n > \frac{\alpha^2}{m^2}$. Taking the

$\rightarrow \sqrt{y_n} > \frac{\alpha}{m}$

$\rightarrow m\sqrt{y_n} > \alpha$. It follows that

$x_n \sqrt{y_n} > \alpha$.

which implies that $\lim\limits_{n \to \infty} x_n \sqrt{y_n} = +\infty$

5. Compute $\lim \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

$(0 < b < a)$

Divide the numerator and denominator by a^n.

We obtain $\dfrac{a + \left(\frac{b}{a}\right)^n b}{1 + \left(\frac{b}{a}\right)^n}$.

If we set $c = \frac{b}{a}$, then

$$\lim_{n \to \infty} \left(\frac{b}{a}\right)^n = \lim_{n \to \infty} c^n = 0.$$

$\therefore$ The limit of the above expression is $\dfrac{a+0}{1+0} = a$.

6. Let $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$.

Show that (x_n) is increasing and bounded, and hence converges

Hint: If $k \geq 2$, then

$$\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k} - \frac{1}{k-1},$$

$x_{k+1} = x_k + \frac{1}{k^2}$, so (x_n) is increasing. (Why?)

$$x_k = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 + \left(\frac{1}{1}\right) - \left(\frac{1}{n}\right) \to 1 + 1 - 0 = 2$$

7 (a) State the Bolzano-Weierstrass Theorem: If I is closed bounded interval, and if (x_n) is a sequence in I, then there is a subsequence (x_{n_k}) that is convergent.

7(b). A sequence (x_n) is Cauchy if for every $\varepsilon > 0$, there is a $K \in \mathbb{N}$, so that if both $m \geq K$ and $n \geq K$, then $|x_m - x_n| < \varepsilon$.