2.5 Intervals

We need to prove a theorem about "nested intervals" before we study 3.4.

We say a sequence of closed bounded $\underbrace{\text{ }}_{}$intervals are **nested** if

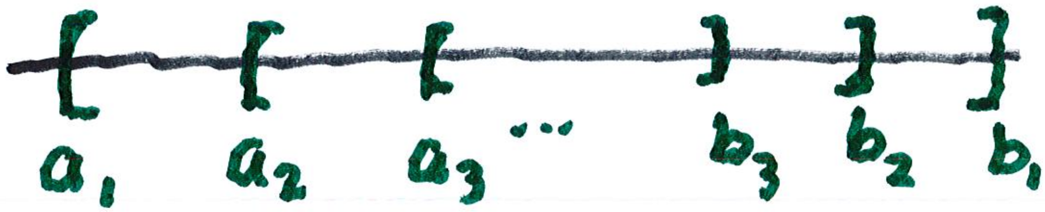
$$I_1 \supseteq I_2 \supseteq \dots \supset I_n \supset I_{n+1} \supset \dots$$

If $I_n = [a_n, b_n]$, then

(b_n) is decreasing, and

(a_n) is increasing, i.e.

we have the picture



We prove the

Nested Interval Property:

Given a sequence of nested closed intervals as above, there is a point η in I_n for all $n \in \mathbb{N}$

Proof. Since $I_n \subseteq I_1$,

we get

$a_n \leq b_n \leq b_1$, for all $n \in \mathbb{N}$.

Hence the sequence (a_n)

is increasing and bounded.

By the Monotone Convergence

Thm., there is an η

satisfying $\eta = \lim(a_n)$.

Clearly $a_n \leq \eta$, all $n \in \mathbb{N}$. (1)

We want to show that

$\eta \leq b_n$ for all n.

We do this by showing that for any particular n,

$b_n \geq a_k$, $k = 1, 2, \ldots$

There are 2 cases.

(i) If $n \leq k$, then since

$I_n \supseteq I_k$, we have

$a_k \le b_k \le b_n.$

(ii) If $k < n$, then since

$I_k \supseteq I_n$, we have

$a_k \le a_n \le b_n$

We conclude that $a_k \le b_n$.

for all k,

so that b_n is

an upper bound for

$\{a_k ; k \in N\}$

Passing to the limit as

k approaches ∞, we obtain

$\eta \le b_n$, for all $n \in N$. (2)

Combining in (1) and (2),

we have

$a_n \le \eta \le b_n$, all $n \in N$.

Hence $\eta \in I_n$ for all n.

3.4. Subsequences

Let $X = (x_n)$ be a sequence

and let

$n_1 < n_2 < \dots < n_k < \dots$

be a strictly increasing

sequence of integers in $\mathbb{N}$.

Then the sequence

$X' = (x_{n_k})$ given by

$(x_{n_1}, x_{n_2}, \dots)$

is called a <u>subsequence</u>

of X.

Ex. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right)$

is a subsequence of

$\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right) = X$

corresponding to $n_k = 2k$.

But $\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \dots\right)$

is not a subsequence of X.

The following theorem is fundamental to the theory of calculus.

Bolzano-Weierstrass Thm.

A bounded sequence of real numbers has a convergent subsequence.

Pf. Since $\{x_n : n \in N\}$ is bounded, this set is contained in an interval $I_1 = [a_1, b_1]$

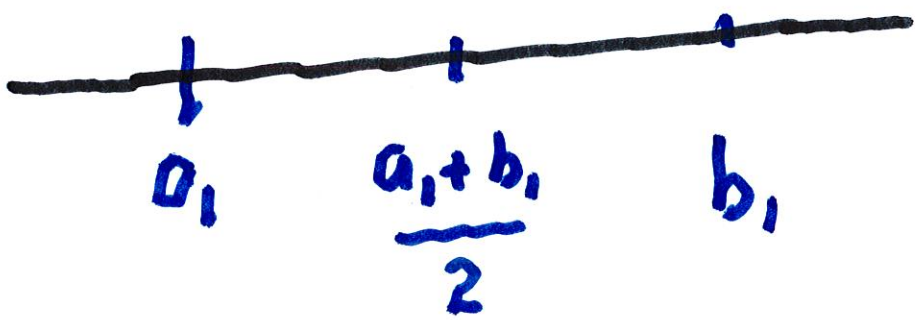
We set $n_1 = 1$.

We now bisect I_1 into two intervals I_1' and I_1''.

More precisely.

$$I_1' = \left[a_1 , \frac{a_1 + b_1}{2} \right] \quad \text{and}$$

$$I_1' = \left[\frac{a_1 + b_1}{2} , b_1 \right]$$



We divide N into two sets.

$$A_1 = \{ n \in N : n > n_1 , x_n \in I_1' \}$$

$$B_1 = \{ n \in N : n > n_1 , x_n \in I_1'' \}$$

If A_1 is infinite, then we set $I_2 = I_1'$, and we let n_2 be the smallest natural number in A_1.

If A_1 is a finite set, then B_1 must be infinite, and we let n_2 be the smallest natural number in N, and we set $I_2 = I_1''$.

We now bisect I_2 into subintervals I_2' and I_2'' and we divide the set

$\{n \in \mathbb{N} : n > n_2\}$ into 2 parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2 , \ x_n \in I_2'\}$$

$$B_2 = \{n \in \mathbb{N} : n > n_2 , \ x_n \in I_2''\}.$$

If A_2 is infinite, we

take $I_3 = I_2'$, and we let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 = I_2''$, and we let n_3 be the smallest natural number in B_2.

We continue in this way to obtain a sequence of nested intervals

$$I_1 \supset I_2 \supset \dots \supset I_k \supset \dots$$

and we obtain a subsequence $\{x_{n_k}\}$ of X such that

$x_{n_k} \in I_k$ for $k \in \mathbb{N}$.

By the Nested Interval Property, there is a point η such that

$$\eta \in \bigcap_{k=1}^{\infty} I_k .$$

The length of I_k is $\frac{(b-a)}{2^{k-1}}$. Since both x_{n_k} and η both lie in I_k, it follows that

$$|x_{n_k} - \eta| \leq \frac{(b-a)}{2^{k-1}},$$

which implies that the subsequence $\{x_{n_k}\}$ of X converges to η.