3.5 Cauchy's Criterion.

Def'n. A sequence $X = (x_n)$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists a number $H(\varepsilon)$ in $\mathbb{N}$ so that if $n, m \geq H(\varepsilon)$, then

$$|x_n - x_m| < \varepsilon$$

Even though the definition does not mention a limit x.

Still, the numbers x_n and x_m get closer as $n, m \to \infty$

Lemma. If a sequence approaches a limit x, then the sequence (x_n) is Cauchy

Proof of Lemma. If $x = \lim(x_n)$, then given $\varepsilon > 0$, there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$, then $|x_n - x| < \frac{\varepsilon}{2}$.

Thus, if $H(\varepsilon) = K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)|$$

$$\leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

it follows that (x_n) is a

Cauchy sequence

Lemma. A Cauchy sequence is bounded.

Pf. Let $X = (x_n)$ be Cauchy, and set $\varepsilon = 1$. If $H = H(1)$, then if $n \geq H$, then

$|x_n - x_H| < 1$. By the Triangle Inequality, we have

$$|x_n| \leq |x_K + (x_n - x_K)|$$

$$\leq |x_K| + 1$$

If we set

$$M = \max \left\{ |x_1|, |x_2|, \dots |x_{K-1}|, |x_H| + 1 \right\},$$

then it follows that

$$|x_n| \leq M, \quad \text{for all } n.$$

Cauchy Convergence Thm.

A sequence $X = (x_n)$ is convergent if it is a Cauchy sequence.

We already showed that if X is convergent, then it is Cauchy. To prove the other direction, suppose X is Cauchy.

We showed above that X is therefore bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence

$X' = (x_{n_k})$ of X that

converges to a number x^*.

We will show that $\lim x_n = x^*$.

Since $X = (x_n)$ is a Cauchy

sequence, given $\varepsilon > 0$, there

is a natural number $H(\varepsilon/2)$

such that if $n, m \geq H(\varepsilon/2)$,

then $|x_n - x_m| < \frac{\varepsilon}{2}$.

Since the subsequence

$X' = \{x_{n_k}\}$ converges to x^*,

there is a natural number

$K \geq H(\varepsilon/2)$ belonging to the

set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \frac{\varepsilon}{2}$$

Since $K \geq H(\varepsilon/2)$, it follows

from ( ) with $m = K$ that

$$|x_n - x_K| < \frac{\varepsilon}{2} \quad \text{for } n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$,

we have

$$|x_n - x^*| = |(x_n - x_K) + (x_K - x^*)|$$

$$\leq |x_n - x_K| + |x_K - x^*|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\lim(x_n) = x^*$.

Ex. The polynomial equation

$x^3 - 5x + 1 = 0$ has a root

r with $0 < r < 1$.

We define an iteration procedure to define a sequence (x_n) that approaches the root r.

We define x_1 to be any number with $0 < x_1 < 1$.

and we define

$$x_n^3 - 5x_{n+1} + 1 = 0,$$

or $$x_{n+1} = \frac{1}{5}(x_n^3 + 1).$$

We can estimate $|(x_{n+2} - x_{n+1})|$

by $|x_{n+2} - x_{n+1}|$

$$= \left| \frac{1}{5}(x_{n+1}^3 + 1) - \frac{1}{5}(x_n^3 + 1) \right.$$

$$= \frac{1}{5}\left| x_{n+1}^3 - x_n^3 \right|$$

$$= \frac{1}{5} \left| x_{n+1}^2 + x_{n+1} x_n + x_n^2 \right| \left| x_{n+1} - x_n \right|$$

$$\leq \frac{3}{5} \left| x_{n+1} - x_n \right|$$

We're using the fact that if $0 \leq x_1 \leq 1$, then x_n also satisfies $0 \leq x_n \leq 1$ for all $n = 1, 2, \ldots$ (by induction)

Hence the sum with 3 terms is in $[0, 3]$.

The above sequence satisfies

$$|x_{n+1} - x_n| \leq \frac{3}{5} |x_n - x_{n-1}|$$

$$\leq \left(\frac{3}{5}\right)^2 |x_{n-1} - x_{n-2}| \leq \dots$$

$$\leq \left(\frac{3}{5}\right)^{n-1} |x_2 - x_1|, \quad \text{for all } n \geq 1.$$

The error difference shrinks

geometrically as $n \to \infty$

The sequence is called

a <u>contractive sequence</u>.

because

$$|x_{n+2} - x_{n+1}| \le C\,|x_{n+1} - x_n|.$$

Thm. Every contractive sequence is a Cauchy sequence.

From ( ) we obtain

$$|x_{n+2} - x_{n+1}| \le C\,|x_{n+1} - x_n|$$

$$\le C^2 |x_n - x_{n-1}| \le \dots \le C^n |x_2 - x_1|$$

More generally, we obtain

$$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| +$$

$$+ |x_{n+1} - x_n|$$

$$\le \left(C^{m-2} + C^{m-1} + \dots + C^{n-1} \right) |x_2 - x_1|$$

$$= C^{n-1} \left(\frac{1 - C^{m-n}}{1 - C} \right) |x_2 - x_1|$$

$$\le C^{n-1} \left(\frac{1}{1 - C} \right) |x_2 - x_1|$$

which shows (x_n) is Cauchy.

We're using the formula

$$C^{m-n-1} + C^{m-n-2} + \dots + 1$$

$$= \frac{1 - C^{m-n}}{1 - C}$$

Back to the proof, we let

$m \to \infty$, and we get

$$|x^* - x_n| \le \frac{C^{n-1}}{1-C} |x_2 - x_1|,$$