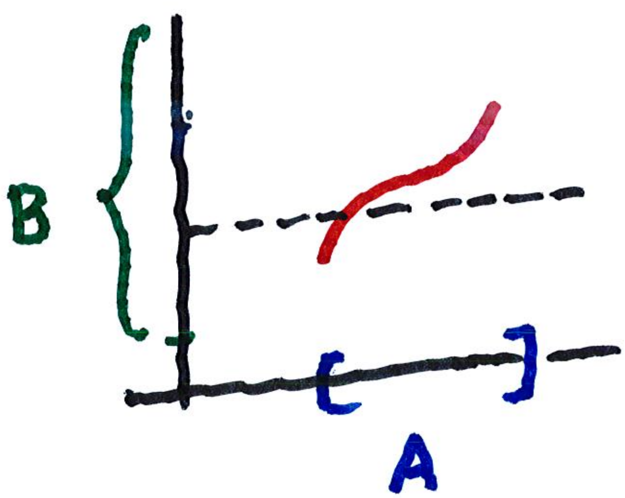
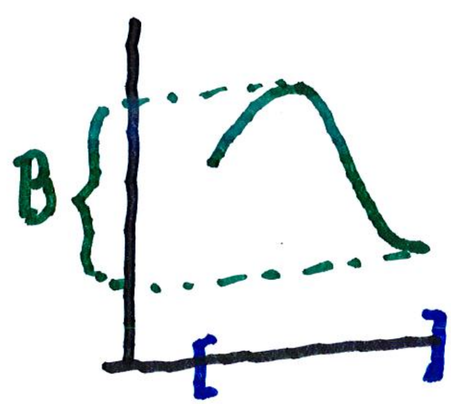
Also $f: A \to B$ is surjective

if whenever $y \in B$, then

there is an x in A so $f(x) = y$

{ f is onto }





f is 1-to-1 but not onto

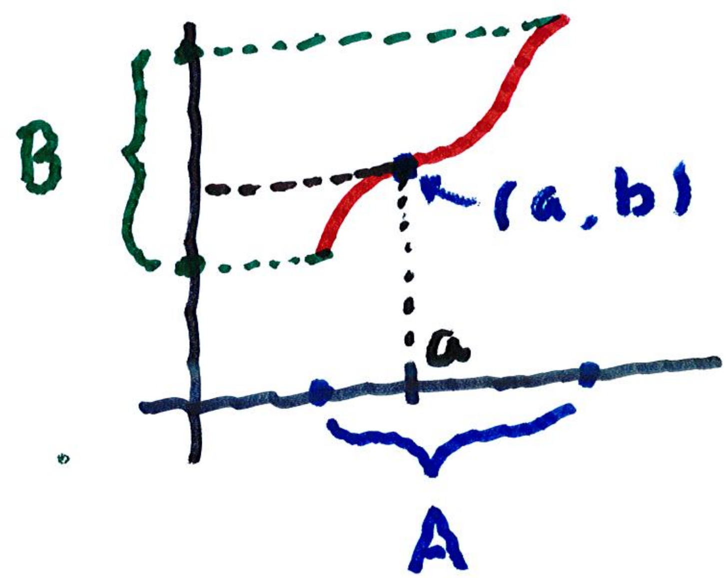
f is onto but not 1-to-1

Lecture 1 cont'd:

We say f is bijective

if f is both injective

and surjective.



$f(a) = b$

$g(b) = a.$

Theorem. Suppose $f : A \longrightarrow B$

is bijective, (i.e., both
onto and
1-to-1)

Then there is a bijection

$g : B \longrightarrow A$ that satisfies

(a) $g(f(a)) = a$ for all
a in A

(b) $f(g(b)) = b$ for all
b in B

First we define $g(b)$ for any b in B

Since f is onto, there is one element a in A so that

(1) $f(a) = b$. Moreover there is only one such a.

For if $\tilde{a} \in A$ with $f(\tilde{a}) = b$, and if $\tilde{a} \neq a$, then this would mean $f(a) = f(\tilde{a})$, which contradicts the fact that f is 1-to-1. Hence we define $g(b) = a$. (2)

Now we show that (b) holds. Apply f to both sides of (2)

$$\Rightarrow \quad f(g(b)) = f(a) = b$$

by (1).

Since b is arbitrary, this proves (b)

Now let a be in A.

Then $b = f(a)$, and as we

saw above, $g(b) = a$. (3)

If we apply g to $\underline{b = f(a)}$,

we get

$$g(b) = g(f(a)).$$

so according to (3)

$$a = g(f(a)). \quad \text{for all } a \text{ in } A.$$

This proves (a).

Finally, note that (a) proves that for any a in A, g maps $f(a)$ to a. $\therefore g$ is onto

Also, if $g(b_1) = g(b_2)$ then (a) shows that

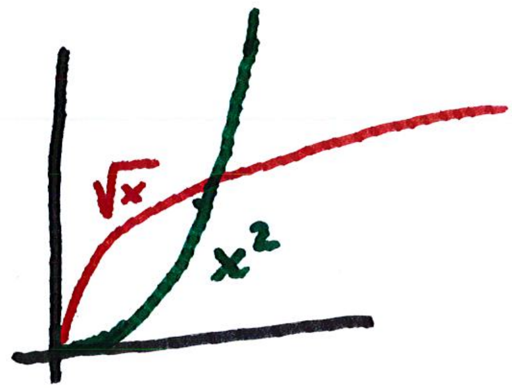
$$f(g(b_1)) = f(g(b_2)),$$

which by (b) implies that $b_1 = b_2$. Hence g is 1-to-1. and so, g is bijective.

Ex. Let $f(x) = x^2$. The $(0 \le x < \infty)$

inverse

of f is $\sqrt{x}$.



$x^2 = 3$

Apply $\sqrt{}$. $\sqrt{x^2} = \sqrt{3}$

or $x = \sqrt{3}$.

Ex. $\sin x$ maps $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1,1]$

$\sin^{-1}$ maps $[-1,1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Suppose $\sin x = .42$

Apply $\sin^{-1}$:

$$\sin^{-1}(\sin x) = \sin^{-1}(.42)$$

$$\rightarrow \quad x = \sin^{-1}(.42)$$

Ex. Let $A = \{x \in \mathbb{R} : x \neq -1\}$

and let $f(x) = \dfrac{2x-1}{x+1}$.

Show that f is injective.

Suppose $f(x_1) = f(x_2)$

$$\frac{2x_1 - 1}{x_1 + 1} = \frac{2x_2 - 1}{x_2 + 1}$$

$$(2x_1 - 1)(x_2 + 1) = (2x_2 - 1)(x_1 + 1)$$

$$2x_1 - x_2 = -x_1 + 2x_2$$

$$\rightarrow \quad 3x_1 = 3x_2$$

or $x_1 = x_2$. ✓

Now find the range of f.

Find all y, such that

$$y = \frac{2x-1}{x+1} \rightarrow yx + y = 2x - 1$$

Solve for x: $(y-2)x = -y-1$

$$\rightarrow \quad x = \frac{y+1}{2-y}$$

This can be solved only

if $y \neq 2$. $R(f) = \left\{ y \in \mathbb{R} : y \neq 2 \right\}$

1.2 Principle of Mathematical Induction

Let S be a subset of $\mathbb{N}$

that satisfies

(1) The number $1 \in S$

(2) For every $k \in \mathbb{N}$,

if $k \in S$, then $k+1 \in S$

Then for all $n \in \mathbb{N}$, $n \in S$.

Note (2) does not ask us to prove that $k \in S$.

We only need to show that "if $k \in S$, then $k+1 \in S$".

Usually, Math. Ind. is used to prove that a sequence of statements are all true. For each $n \in N$, let $P(n)$ be a meaningful statement about $n \in N$. We let

$$S = \{ n \in N ; P(n) \text{ is true.} \}$$

The above Mathematal Induction Principle becomes:

Suppose that

(1') $P(1)$ is true.

(2') For every $k \in N$, if

$P(k)$ is true, then

$P(k+1)$ is true.

Then $P(n)$ is true for all $n \in N$.

Ex. Suppose $P(n)$ is the statement that $n^2 - 3n + 2 = 0$.

Note that $P(1)$ is true,

because $1^2 - 3 \cdot 1 + 2 = 0$

But it is not true that

if $P(k)$ is true, then $P(k+1)$ is true.

In fact, if $k = 2$, then $P(2)$ is true. But, $P(3)$ is false.

Ex. Suppose $P(n)$ is the statement that

$f(n) = n^2 - n + 41$ is prime.

Note that when $n=1$,

$1^2 - 1 + 41$ is prime.

Then $P(1)$ is true.

But (after some calculation)

$f(40) = 1601$ is prime and

$f(41) = 41^2 = 1681$

$\therefore$ $f(41)$ is NOT prime.

Hence $P(40)$ is true but

$P(41)$ is false.

Thus (2) fails when $n = 40$

Ex. Use Math. Ind. to prove

that

$$1^2 + 2^2 + 3^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$$

When $n=1$, $P(1)$ is the statement

$$1^2 = \frac{1 \cdot 2 \cdot (3)}{6} = 1$$

$\therefore$ $P(1)$ holds.

Now suppose $P(k)$ is true.

Then

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

by the induction assumption

Now check $P(k+1)$:

$$1^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6k + 6\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + 7k + 6\right]$$

$$= \frac{(k+1)}{6}(k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$\therefore$ P(k+1) is true.

$\therefore$ (2) holds

Since both (1) and (2) are true, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

Hence

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Ex. Prove that $5^{2n}-1$ is divisible by 8. This is $P(n)$.

(1) When $n=1$,

$5^2-1 = 24 = 3\cdot 8$,

so $P(1)$ is true.

(2) Suppose that $P(k)$ is true,

i.e., $5^{2k}-1$ is divisible by 8.

Check P(k+1):

$$5^{2(k+1)} - 1 = 5^2 \cdot 5^{2k} - 1$$

$$= 5^2(5^{2k} - 1) + 5^2 - 1$$

$$= 5^2(5^{2k} - 1) + (5^2 - 1)$$

Both are divisible by 8

$\therefore$ P(k+1) is true.

$\rightarrow$ (2) holds $\Rightarrow$ $5^{2n} - 1$ is div. by 8 for all n.

Bernoulli's Inequality

Show that

for all $n \in \mathbb{N}$ and for all $x > -1$,

$$(1+x)^n \geq (1+nx)$$

Pf. First we check $P(1)$

$$(1+x)^1 = (1+1\cdot x) \quad \checkmark.$$

Now check (2)

Suppose that $(1+x)^k \geq 1+kx$

for all $x > -1$.

Note that

$$(1+x)^{(k+1)} = (1+x)^k (1+x)$$

$$\geq (1+kx)(1+x)$$

by the inductive hypothesis

and that $1+x > 0$.

$$= 1 + kx + x + kx^2$$

$$\geq 1 + (k+1)x.$$

Thus $P(k+1)$ is true,

and hence (2) holds.

By induction, $P(n)$ is true

for all $n \in N$

$$\Rightarrow (1+x)^n \geq 1 + nx, \text{ when } x > -1.$$

Sometimes, the statement is only defined for $n \geq n_0$

Modified Principle of Math. Induction.

Suppose that

(1) $P(n_0)$ is true.

(2) For all $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$

Ex. Prove that

$2^n < n!$ for all $n \geq 4$.

Note that when $n = 4$,

$$2^4 = 16 < 24 = 4!$$

This shows that $P(4)$ holds.

Now let k be an integer ≥ 4, and assume that

$2^k < k!$

Note that since $k \geq 4$

$2^{(k+1)} = 2^k \cdot 2 < (k!)2$

$< (k!)(k+1) = (k+1)!$

$\uparrow$

Since $2 < k+1$.

Hence $P(k+1)$ is true. By

induction $P(n)$ is true for

all $n \geq 4$.

Sometimes the Induction Principle can be expressed as follows.

Let S be a subset of $\mathbb{N}$ such that

(1) $P(1)$ is true.

(2) For every $k \in \mathbb{N}$,

if $P(1), \ldots, P(k)$ are all true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

(This is sometimes called the Principle of Strong Induction.)

Ex. Suppose a sequence

$\{x_n\}$ is defined by

$x_1 = 1, \quad x_2 = 2$ and

$$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n).$$

Use Strong Induction to

show that

$$1 \le x_n \le 2, \quad \text{all } n \in \mathbb{N}.$$

Let $P(n)$ be the statement that $1 \le x_n \le 2$.

Note that $P(1)$ and $P(2)$ both hold by hypothesis.

Now let $k \in \mathbb{N}$ with $k \ge 2$, and suppose that $P(j)$ is true for all $j \le k$.

Then $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\leq \frac{1}{2}(2+2) = 2$

by strong induction hypothesis

and

$x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\geq \frac{1}{2}(1+1) = 1$

by strong ind. hypothesis

Hence $1 \le x_{k+1} \le 2$,

which shows that $P(k+1)$

is true. Thus the Strong

Induction Principle

implies that $P(n)$ is

true for all $n \in \mathbb{N}$.