6.4 Taylor Series.

Suppose we write a polynomial p

as $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

Set $x=0 \longrightarrow \underline{p(0) = a_0}$

Diff:

$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}$

Set $x=0 \rightarrow \underline{p'(0) = a_1}$

Diff:

$p''(x) = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \dots$

Set $x=0$ $\underline{p''(0) = 2\cdot a_2}$

Keep going: we get

$$p^{(k)}(0) = k(k-1)\ldots 2\cdot 1\, a_k$$

Solve for a_k:

$$a_k = \frac{p^{(k)}(0)}{k!}$$

If we write

$$p(x) = a_0 + a_1(x-a) + a_2(x-a)^2$$

$$\ldots + a_n(x-a)^n,$$

then $a_k = \dfrac{p^{(k)}(a)}{k!}$

Now suppose that f is a function (not nece sarily a polynomial) such that

$f^{(1)}(a), \ldots, f^{(n)}(a)$ all exist.

We define $a_k = \dfrac{f^{(k)}(a)}{k!}$ and

$$P_{n,a}(x) = a_0 + a_1(x-a) + \ldots a_n(x-a)^n$$

$P_{n,a}$ is the n-th Taylor polynomial of degree n for f at a.

The Taylor polynomial has been defined so that

$$P_{n,a}^{(k)}(a) = f^{(k)}(a) \text{ for } 0 \le k \le n.$$

It's the only polynomial of degree $\le n$ with this property.

Ex. Let $f(x) = \sin x$

$\sin 0 = 0$

$\sin'(0) = \cos 0 = 1$

$\sin''(0) = 0$

$\sin^{(3)}(0) = -\cos 0$
$= -1$

$\sin^{(4)}(0) = \sin 0 = 0$

From this point on, the derivatives repeat in a cycle of 4.

→ $a_k = \dfrac{\sin^{(k)}(0)}{k!}$

→ The Taylor polynomial $P_{2n+1,0}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Ex. Consider $f(x) = e^x$.

Since $f^{(n)}(0) = 1$ for all n,

we obtain

$$p_{n,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Ex. For $f(x) = \log x$, use $a = 1$.

$f(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad \log'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad \log''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad \log'''(1) = 2$

In general,

$$\log^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k},$$

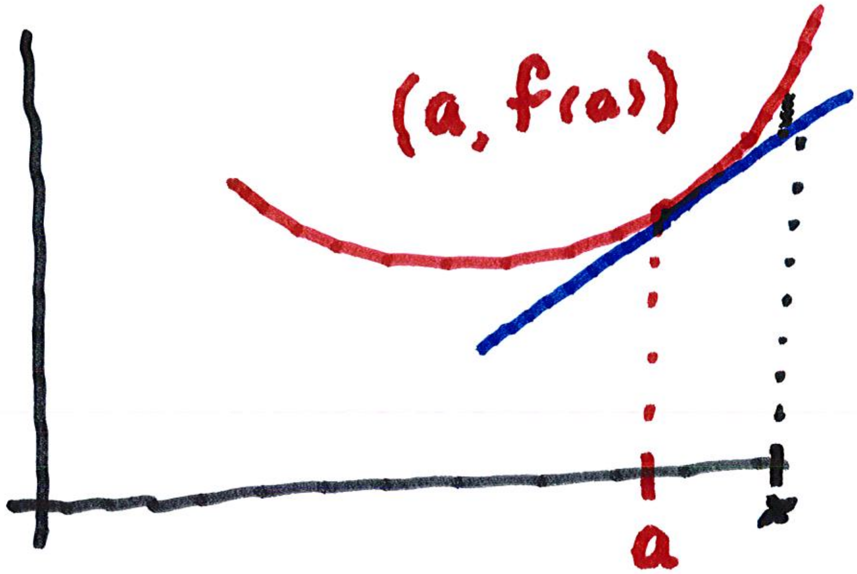
so $\log^{(k)}(1) = (-1)^{k-1}(k-1)!$

$\therefore P_{n,1}(x)$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}$$

Clearly $y = f(a) + f'(a)(x-a)$

is equal to the tangent line



The error = $|f(x) - P_{1,a}|$ is

smaller than $|x-a|$.

We will see that

$|f(x) - P_{n,a}(x)|$ is much smaller

than $|x-a|^n$

We want a formula for the error:

$$R_n(x) = f(x) - P_n(x)$$

Part 2 of the Fundamental Thm. of Calculus is

$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

We integrate by parts:

$u = f'(t)$ $\quad dv = 1 \cdot dt$

$du = f''(t)\,dt$ $\quad v = t - x$

$$= f'(t)(t-x)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt$$

$$= f'(x)\cdot 0 - f'(a)(a-x) + \int_a^x f''(t)(x-t)\,dt$$

Hence,

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt$$

By repeatedly increasing by parts, we get

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + \dots + f^{(n)}(a)\frac{(x-a)^n}{n!} + R_n(x), \quad \text{where}$$

$$R_n(x) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$$

The above formula for $R_n(x)$

is called the "<u>integral</u> <u>form</u>"

of the error.

In order to estimate it,

set $M_{n+1} = \sup\limits_{t \in [a,x]} \{ |f^{(n+1)}(t)| \}$.

This gives

$$|R_n(x)| \le \int_a^x |f^{(n+1)}(t)| \frac{(x-t)^n}{n!} \, dt$$

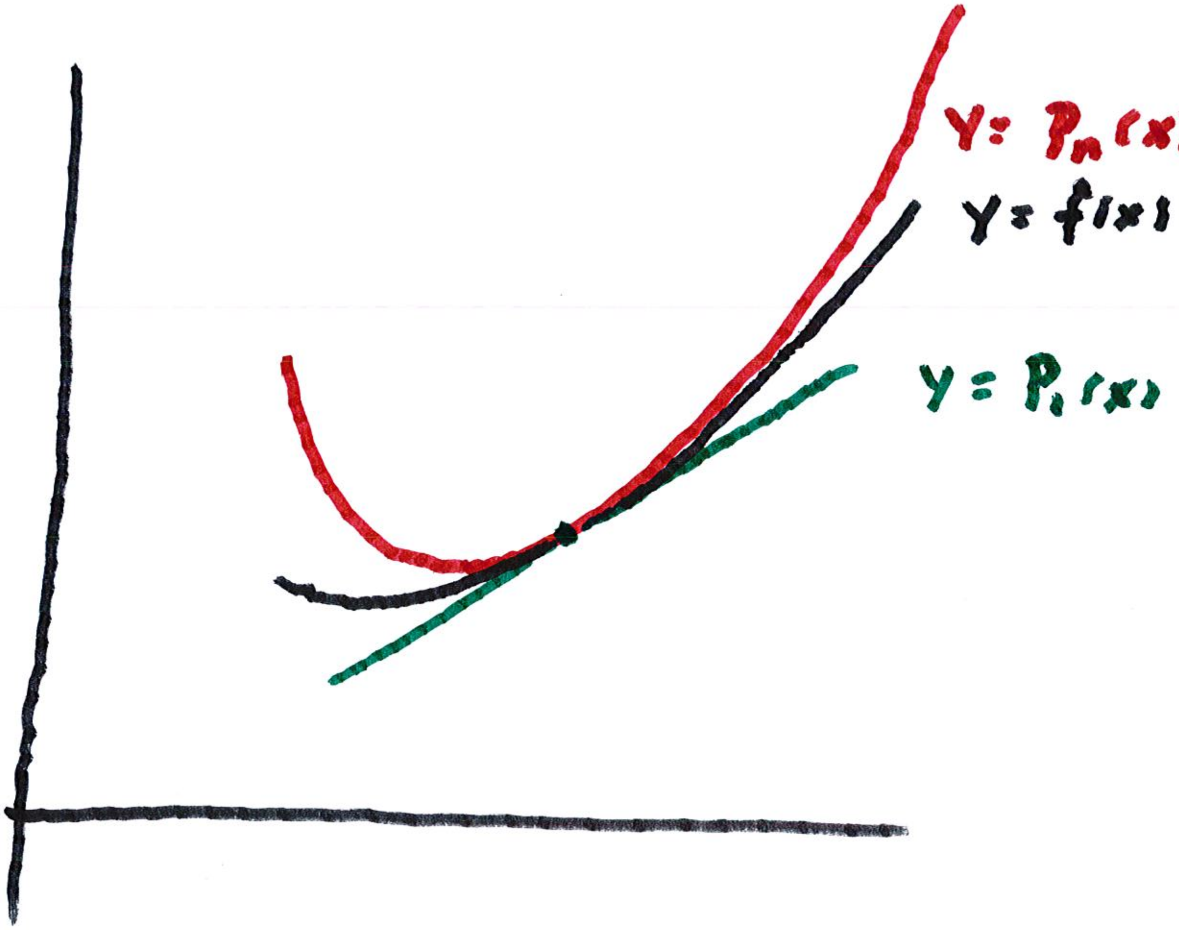
$$\le M_{n+1} \int_a^x \frac{(x-t)^n}{n!}$$

$$= -M_{n+1} \left. \frac{(x-t)^{n+1}}{(n+1)!} \right|_{t=a}^{t=x}$$

$$= M_{n+1} \frac{|x-a|^{n+1}}{(n+1)!}$$

Thus, we've showed

$$|R_n(x)| \le \frac{M_{n+1} |x-a|^{n+1}}{(n+1)!}$$



Note that ~~P_n~~ if x is close to a, then $|x-a|^{n+1}$ is much smaller than $|x-a|^2$.

Let $P_{2N+1}(x) = x - \frac{x^3}{3!} \cdots \pm \frac{x^{2N+1}}{(2N+1)!}$

converges to $\sin x$ as $N \to \infty$.

In fact, note that ~~the~~

$$|\sin^{(n)} x| \leq 1 \quad .$$

Hence $|R_{2N+1}(x)| \leq \frac{1 \cdot |x|^{2N+2}}{(2N+1)!}$

$\to 0$ as $N \to \infty$ (for fixed $|x|$)

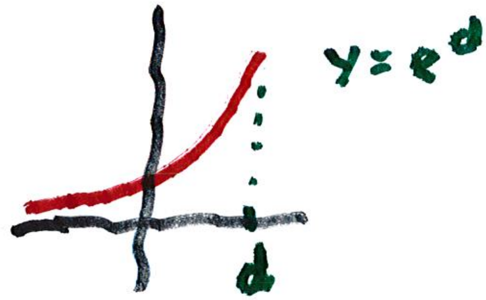
(Use the Ratio Test.)

$$\therefore \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$$

For $f(x) = e^x$, note that

$|f^{(n+1)}(x)| \le e^x \le e^d$ if

$|x| \le d$.



Hence $R_n(x) \le \dfrac{e^d \cdot d^{n+1}}{(n+1)!} \to 0$

by the Ratio Test.

$$\therefore\ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$